Trajectories

MS4414 Theoretical Mechanics

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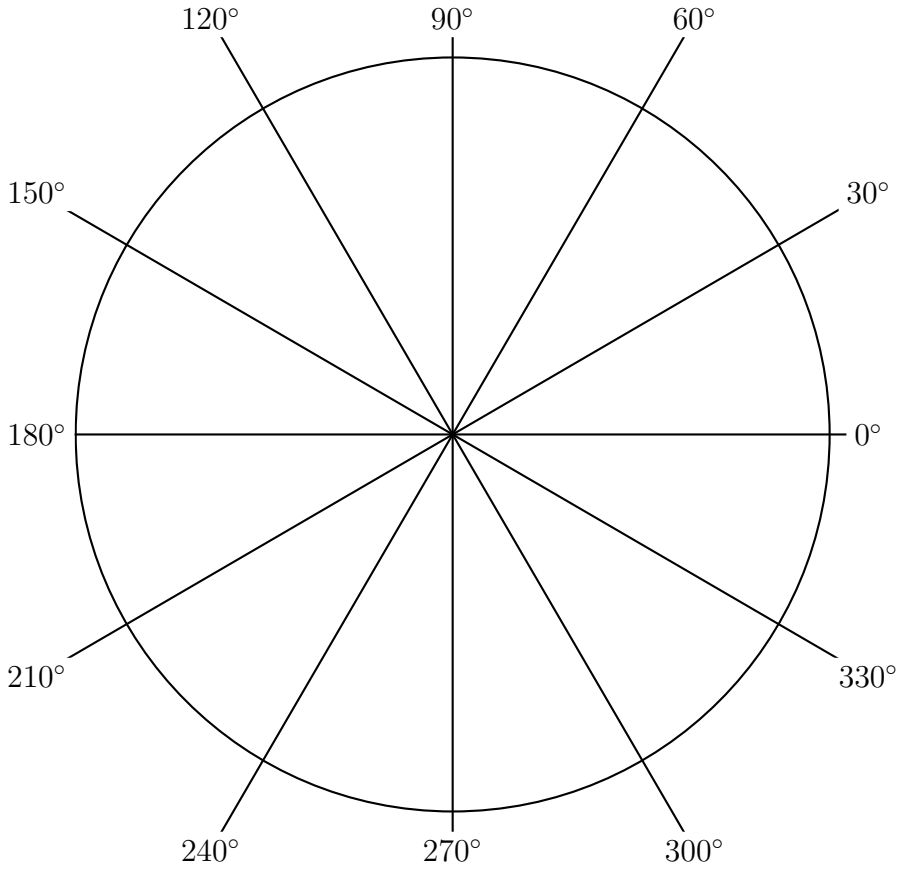
Contents

1 Degrees and Radians

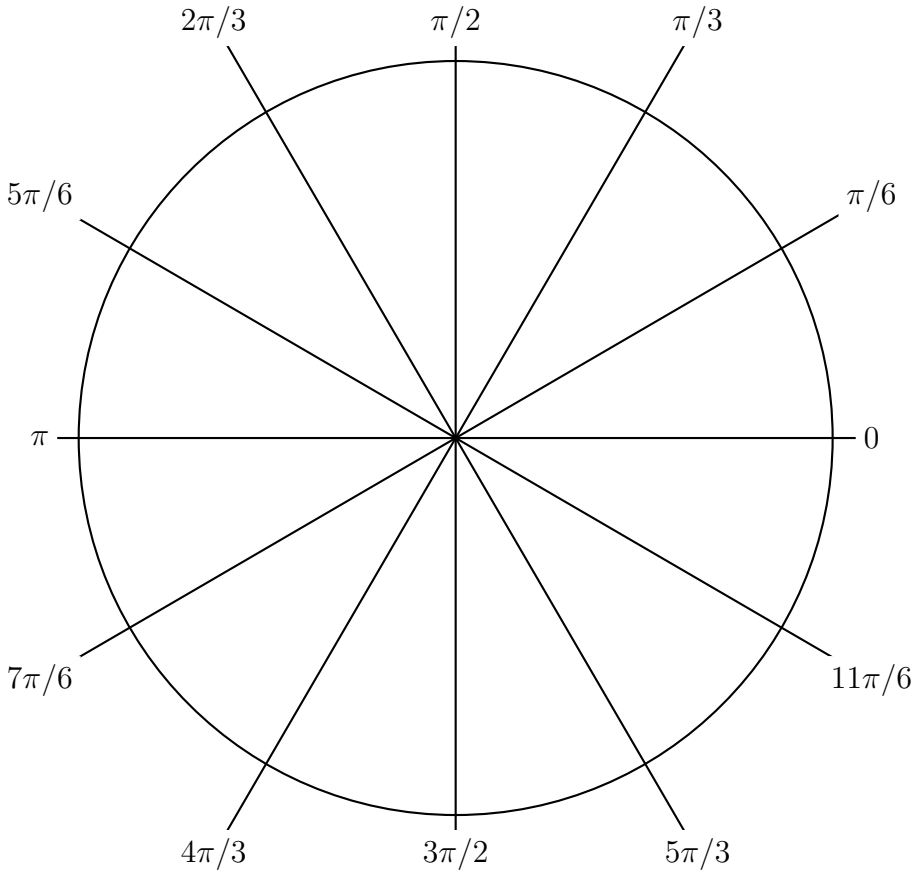
There are (at least) two different ways of measuring angles: using degrees and using radians.

Khan Academy
Radians and degrees.

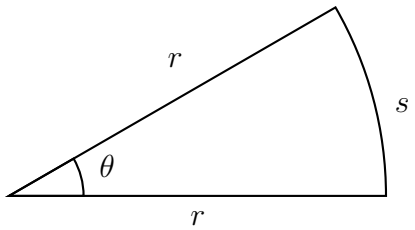
Degrees There are 360 degrees in a circle.



Radians There are $\boxed{2\pi}$ radians to a circle.



Radians and Arcs Radians can be defined as arc length divided by radius.



$$\theta = \boxed{\frac{s}{r}}$$

Radians and Calculus Radians are natural units for calculus. If θ is measured in radians

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta = \cos\theta$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta}\cos\theta = \boxed{-\sin\theta}$$

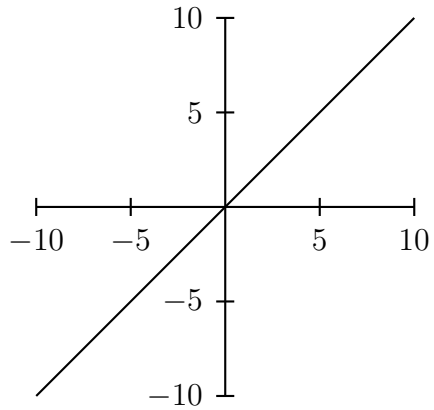
Worked Example (a) Translate the following angles in degrees into radians: (i) 1°; (ii) 45°; (iii) 60°. (b) Translate the following angles in radians into degrees: (i) $\frac{\pi}{4}$; (ii) $\frac{\pi}{10}$; (iii) 1.

Degrees	Radians
1°	0.017453
45°	0.78540
60°	1.0472
45°	$\frac{\pi}{4}$
18°	$\frac{\pi}{10}$
57.296°	1

Exam Technique Always check whether your calculator is in degrees or radians mode before starting a calculation involving angles, or if you get a result that doesn't make sense.

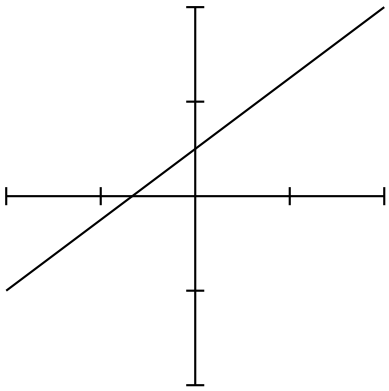
2 Graphs of Functions

2.1 Straight Lines



$f(x) = x$

x	$f(x)$
-10	-10
-5	-5
0	0
5	5
10	10

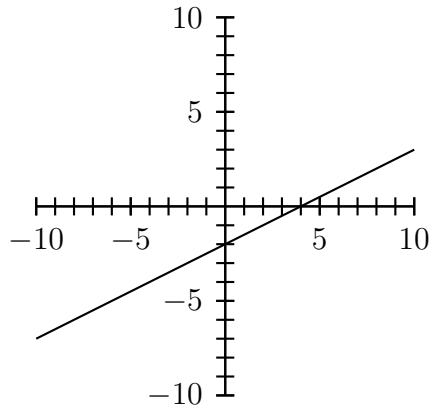


$f(x) = ax + b$

gradient $= a$

y-intercept $= b$

x-intercept $= -b/a$



$f(x) = 0.5x - 2$

x	$f(x)$
-10	-7.0
-5	-4.5
0	-2.0
5	0.5
10	3.0

2.2 Trigonometric Functions

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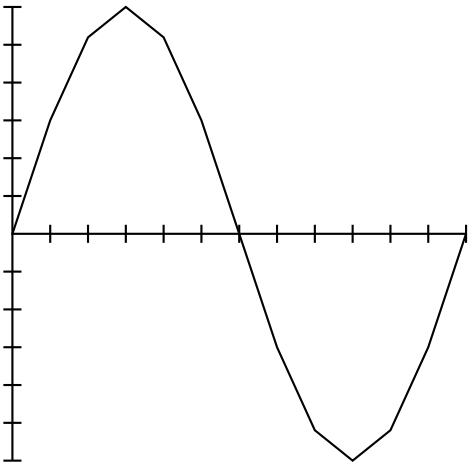
Graph of sine.

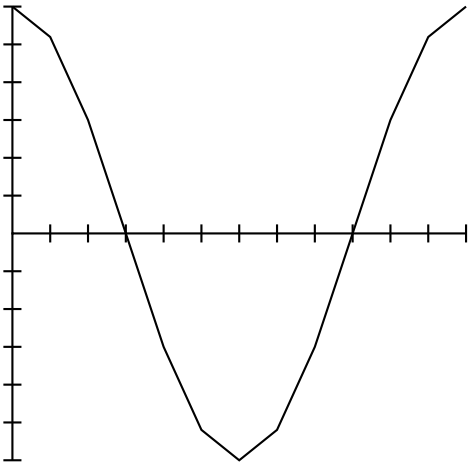
$f(x) = \sin x$



x	$f(x)$
0	0.00000
$\pi/6$	0.50000
$2\pi/6$	0.86603
$3\pi/6$	1.00000
$4\pi/6$	0.86603
$5\pi/6$	0.50000
$6\pi/6$	0.00000
$7\pi/6$	-0.50000
$8\pi/6$	-0.86603
$9\pi/6$	-1.00000
$10\pi/6$	-0.86603
$11\pi/6$	0.50000
$12\pi/6$	0.00000

Khan Academy
Trig graphs.

$f(x) = \cos x$



x	$f(x)$
0	1.00000
$\pi/6$	0.86603
$2\pi/6$	0.50000
$3\pi/6$	0.00000
$4\pi/6$	-0.50000
$5\pi/6$	-0.86603
$6\pi/6$	-1.00000
$7\pi/6$	-0.86603
$8\pi/6$	-0.50000
$9\pi/6$	0.00000
$10\pi/6$	0.50000
$11\pi/6$	0.86603
$12\pi/6$	1.00000

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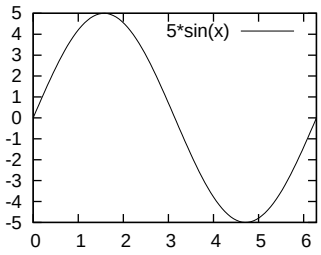
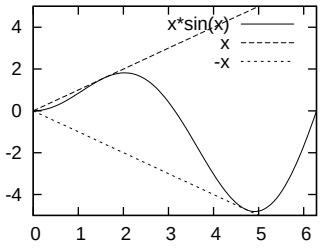
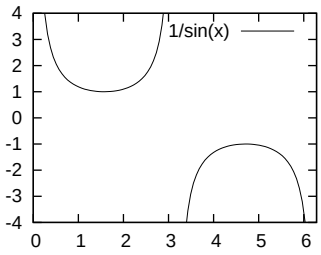
Trig graphs II.

Worked Example Draw the graphs of $f(x) = 5\sin x$, $f(x) = x\sin(x)$ and $f(x) = 1/\sin x$







Exam Technique In an exam there is no time to work out tables of values, and such tables may miss important points like asymptotes. You must know the curves well enough to sketch them almost at once. If the curve is unfamiliar, try to see if it is related to a familiar curve.

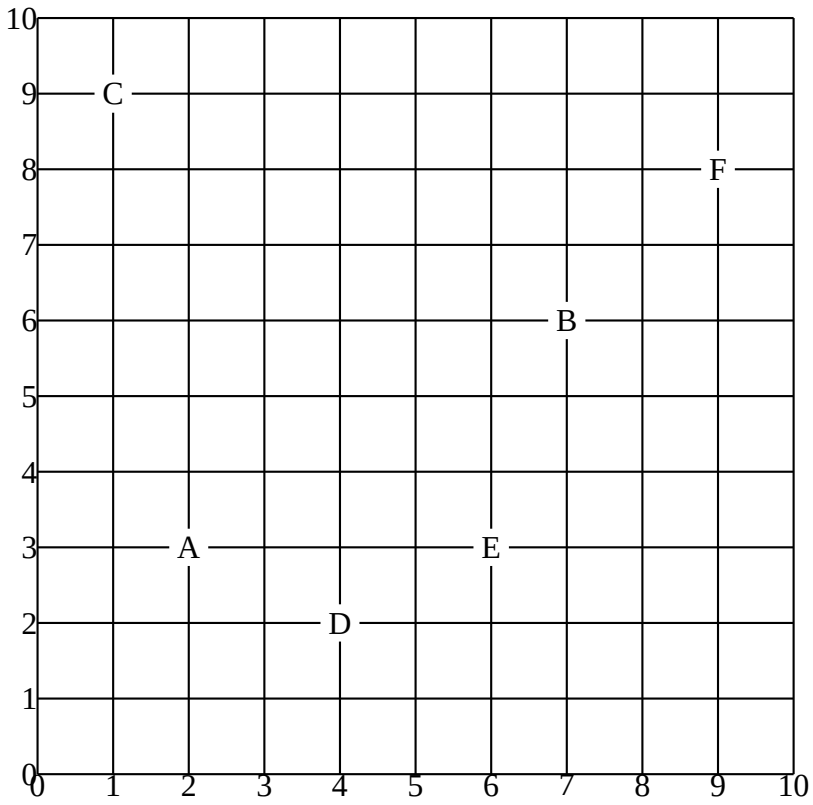
Curves with envelopes e.g. $x^2\cos x$ Draw the envelope in first and then sketch the curve inside it.

Function of a known curve e.g. $\cos^2 x$ Draw this curve below and then transform the special points.

(You may get marks for these intermediate steps even if there is a mistake in the final curve.)

3 Cartesian and Polar Coordinates

3.1 Cartesian coordinates



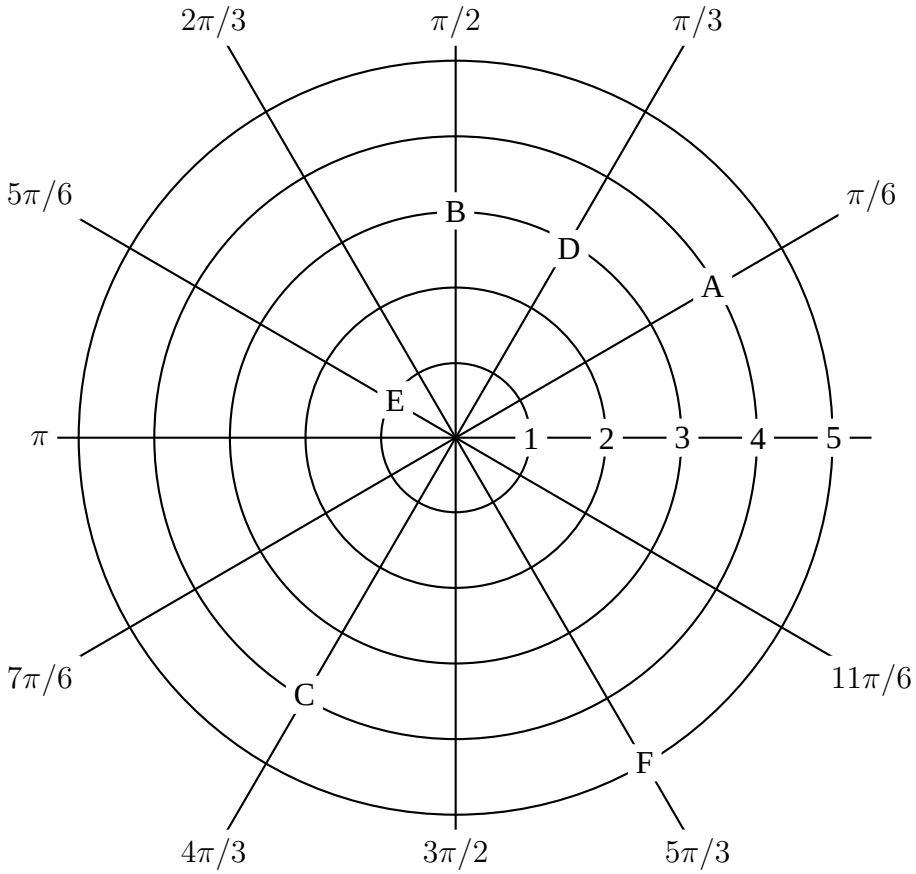
The coordinates of the points A, B and C are:

A $(2, 3)$; B $(7, 6)$; C $(1, 9)$.

Points D, E and F are at: D $(4, 2)$; E $(6, 3)$; F $(9, 8)$.

Khan Academy
Polar Coordinates.

3.2 Polar coordinates



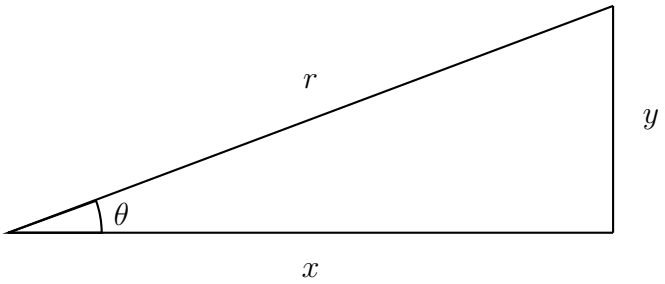
The points A, B and C are at (polar) coordinates: A $\left(4, \frac{\pi}{6}\right)$; B $\left(3, \frac{\pi}{2}\right)$; C $\left(4, \frac{4\pi}{3}\right)$. The points D, E, and F are at (polar) coordinates: D $\left(3, \frac{\pi}{3}\right)$; E $\left(1, \frac{5\pi}{6}\right)$; F $\left(5, -\frac{\pi}{3}\right)$.

3.3 Conversion

Khan Academy
Curves in Polar

Coordinates.

Relation between Cartesian and polar coordinates.



$$x(r,\theta) = \boxed{r\cos\theta}$$

$$y(r,\theta) = \boxed{r\sin\theta}$$

$$r(x,y) = \boxed{\sqrt{x^2+y^2}}$$

$$\theta(x,y) = \boxed{\arctan\left(y/x\right)}$$

Warning Note that $\arctan$ has *two* solutions

$$y/x = -y/-x \implies \tan\theta = \tan\left(\theta+\pi\right).$$

If you use a calculator, do a quick mental check that the answer is in the quarter you expect (based on the signs of x and y). Many computer programs include a function `atan2(y,x)` which give the correct angle.

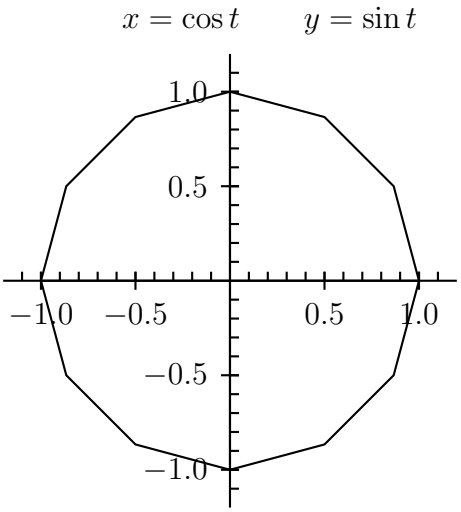
Worked Example Translating between polar and Cartesian coordinates

x	y	r	θ
2	3	3.6056	0.98279
7	6	9.2195	0.70863
1	9	9.0554	1.46014
3.4641	2.0000	4	$\pi/6$
0.0000	3.0000	3	$\pi/2$
–4.0000	0.0000	4	$4\pi/3$

Khan Academy Parameteric Equations I.

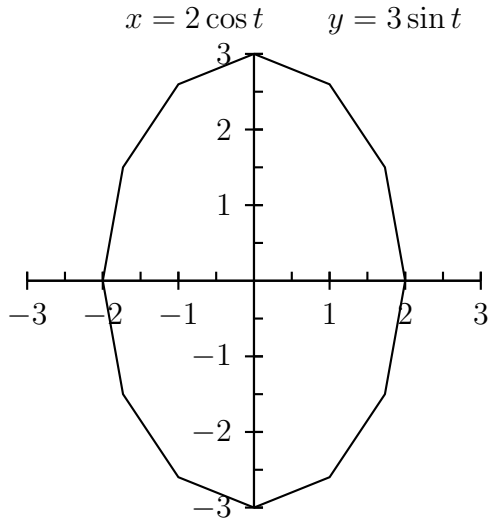
4 Parametric Trajectories in Cartesian Coordinates

A *parametric* curve is one in which all coordinates depend on an additional *parameter*: e.g. x and y coordinates depend on time t.

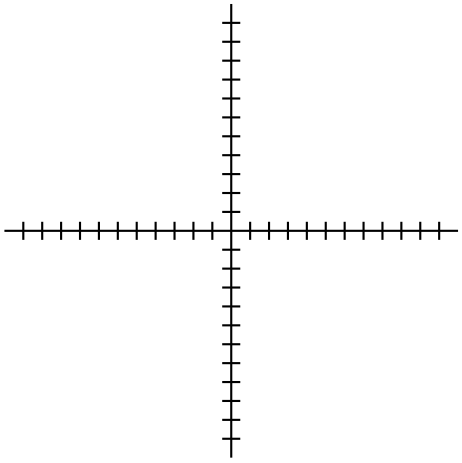


t	x	y
$0\pi/6$	1.000000	0.000000
$1\pi/6$	0.866025	0.500000
$2\pi/6$	0.500000	0.866025
$3\pi/6$	0.000000	1.000000
$4\pi/6$	−0.500000	0.866025
$5\pi/6$	−0.866025	0.500000
$6\pi/6$	−1.000000	0.000000
$7\pi/6$	−0.866025	−0.500000
$8\pi/6$	−0.500000	−0.866025
$9\pi/6$	−0.000000	−1.000000
$10\pi/6$	0.500000	−0.866025
$11\pi/6$	0.866025	−0.500000
$12\pi/6$	1.000000	−0.000000

Khan Academy Parametric equations.



t	x	y
$0\pi/6$	2.000000	0.000000
$1\pi/6$	1.732051	1.500000
$2\pi/6$	1.000000	2.598076
$3\pi/6$	0.000000	3.000000
$4\pi/6$	−1.000000	2.598076
$5\pi/6$	−1.732051	1.500000
$6\pi/6$	−2.000000	0.000000
$7\pi/6$	−1.732051	−1.500000
$8\pi/6$	−1.000000	−2.598076
$9\pi/6$	−0.000000	−3.000000
$10\pi/6$	1.000000	−2.598076
$11\pi/6$	1.732051	−1.500000
$12\pi/6$	2.000000	−0.000000



$$x = a\cos t \qquad y = b\sin t$$

+ve x-intercept $= a$

+ve y-intercept $= b$

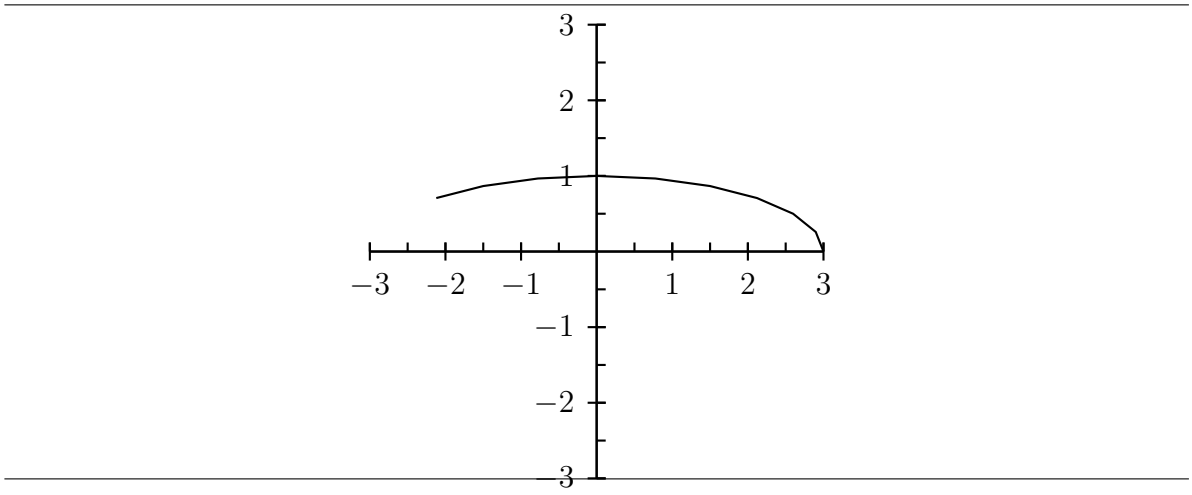
−ve x-intercept $= -a$

−ve y-intercept $= -b$

Exam Question, 2007 paper. The trajectory of a particle is given by

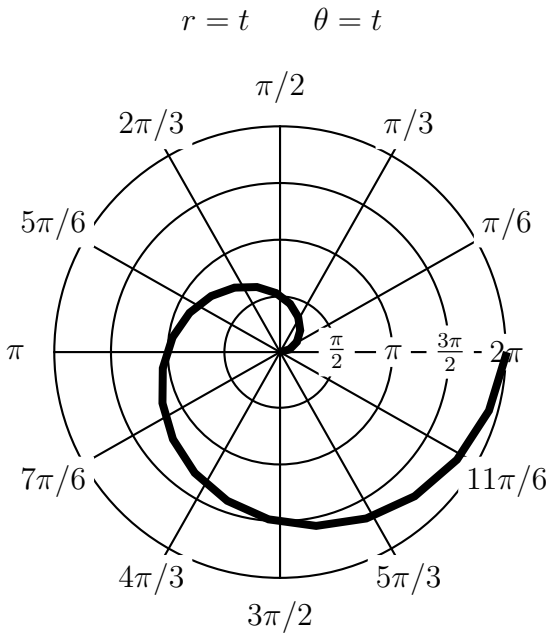
$$x = 3\cos t \qquad y = \sin t \qquad t : 0 \to \frac{3\pi}{4}$$

Sketch the trajectory of the particle.

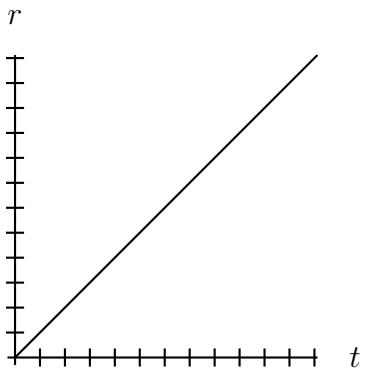


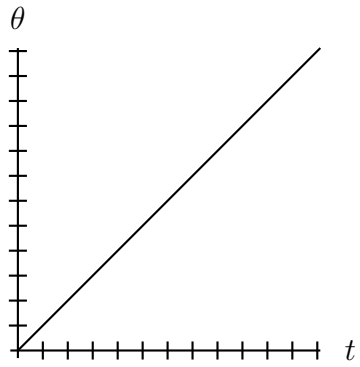
Exam Technique Note that the question only asks for the curve to be sketched for a limited range of t. Label special points (intercepts with axes, asymptotes), and it is worth giving t values as well as x and y values. Consider sketching $x(t)$ and $y(t)$: these sketches may get you intermediate results even if there is a mistake in the final curve.

5 Parametric Trajectories in Polar Coordinates



t	r	θ
0		
$\pi/6$		
$2\pi/6$		
$3\pi/6$		
$4\pi/6$		
$5\pi/6$		
$6\pi/6$		
$7\pi/6$		
$8\pi/6$		
$9\pi/6$		
$10\pi/6$		
$11\pi/6$		
$12\pi/6$		

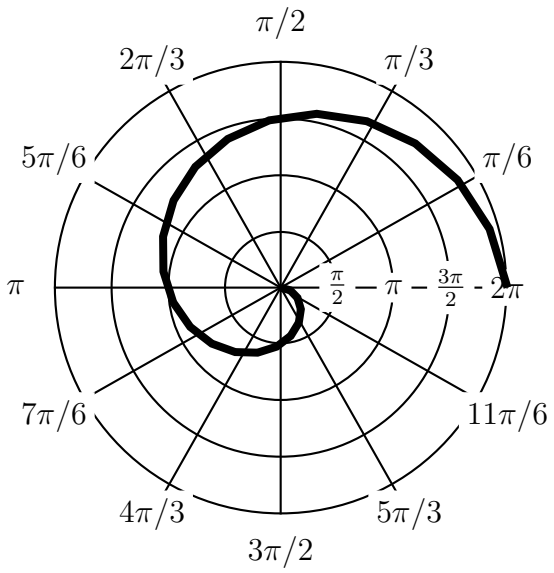




Exam Question 2007 The trajectory of a particle in polar coordinates is given by

$$r = 2\pi - t, \qquad \theta = t, \qquad t:\ 0 \to 2\pi$$

Sketch the trajectory of the particle.



Microsoft Hiring Question Microsoft interviews often involve complex logic puzzles or open ended questions. (This practise has been adopted by many other companies inside and outside the software industry.)
How many points on the Earth are there where by walking one mile south, one mile east and one mile north you will reach the place you started?

1. At the north pole. (One point.)
2. One mile north of the point close to the south pole where the line of latitude is one mile long. (Infinity of real numbers points.)
3. One mile north of the point close to the south pole where the lines of latitude are $\frac{1}{n}$ miles long. (Infinity of real numbers $\times$ infinity of natural numbers points.)