Newton's Laws of Motion

MS4414 Theoretical Mechanics

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First Law In the absence of external forces, a body moves in a straight line with constant velocity.

$$\mathbf{F} = \mathbf{0} \implies \mathbf{v} = \mathbf{const}.$$

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Newton I.

Second Law The rate of change of momentum of a body is equal to the net force on the body.

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

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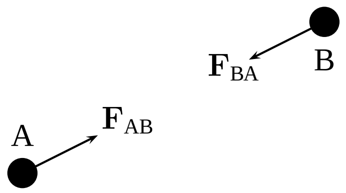
Newton II.

Third Law If body A exerts a force on body B then body B exerts an equal and opposite force on body A.

$$\mathbf{F}_{\mathrm{AB}} = -\mathbf{F}_{\mathrm{BA}}$$



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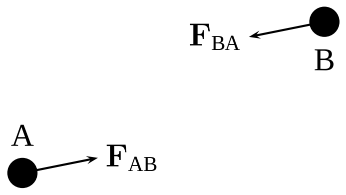
Newton III.

Note that Newton's third law does not require the equal and opposite forces to act along the same line of action (e.g. two charged particles interacting through the magnetic field).



1 Forces

A force is a push or a pull on a body. The units of force are newtons, N. In this course we will encounter the following forces:

Contact forces These are forces which prevent two bodies occupying the same space at the same time. (Ultimately they result from the Pauli exclusion principle of quantum mechanics which states that two particles cannot possess the same set of quantum numbers.)

Friction forces These forces prevent or oppose the motion of a body.

Gravitational forces Attractive forces between two bodies by virtue of their mass.

Elastic forces These forces arise as deformed bodies attempt to recover their original shape.

2 The first and second laws

Newton's first and second laws can be written together as a single equation

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) \tag{1}$$

where $\mathbf{F}$ is the imposed force, m is the mass of the body and $\mathbf{v}$ is its velocity. The vector $m\mathbf{v} = \mathbf{p}$ is called the *momentum*. *If and only if* the mass of the body is constant (not the case

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Momentum.

for a rocket for example) this equation can be rewritten as

$$\mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \tag{2}$$

$$\mathbf{F} = m\mathbf{a} \tag{3}$$

Worked Example Show that Newton's first law of motion can be derived from the equation $\mathbf{F} = m\mathbf{a}$.

Newton's first law follows from $\mathbf{F} = 0$. Then

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 0.$$

Divide by m and integrate

$$\mathbf{v} = \textbf{const.}$$

In the absence of forces the body moves in a straight line with constant velocity.

Worked Example What is the trajectory $\mathbf{x}(t)$ of a particle of mass, m, subject to a constant force $\mathbf{F}$?

Newton's second law

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

where $\mathbf{p} = m\mathbf{v}$ is momentum, and m is constant.

$$\mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}$$

Integrate twice

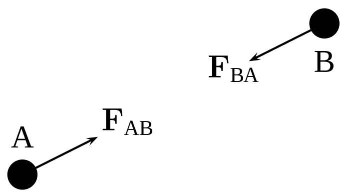
$$\mathbf{x} = \frac{\mathbf{F}}{2m}t^2 + \mathbf{v}_0 t + \mathbf{x}_0$$

3 Newton's Third Law and Conservation of Momentum

N.B. proving the law of conservation of momentum (or angular momentum) often comes up in exams.

Newton's third law states that if body A exerts a force $\mathbf{F}_{\text{BA}}$ on body B, then the force exerted by body B on body A, $\mathbf{F}_{\text{AB}}$, is equal in magnitude and opposite in direction i.e.

$$\mathbf{F}_{\text{BA}} = -\mathbf{F}_{\text{AB}}.$$



Theorem Conservation of momentum. If a collection of particles interact with each other but not with any external entities, the total momentum

$$\sum \mathbf{p}_i = \sum_i \boxed{m_i \mathbf{v}_i}$$

of the collection is independent of time.

Proof 1 (with words)

- Newton's second law of motion states that the rate of change of the momentum of a particle is equal to the total [force] acting on it.
- Therefore the rate of change of the total momentum of all the particles in the collection is the total [force] exerted on all the particles.
- Newton's third law states that forces come in [equal] and [opposite] pairs.
- Therefore the total force acting on the collection of particles is [zero].
- Therefore the rate of change of the momentum of the collection of particles is [zero].
- Therefore the momentum is a (vector) constant.

Proof 2 If there is a collection of particles i, ($i = 1 \ldots n$) interacting with each other but not with any external particles. If $\mathbf{F}_{ij}$ is the force exerted *on* particle i *by* particle j. The rate of change of the total momentum of particle i is given by

$$\boxed{\sum_{j=1}^{n} \mathbf{F}_{ij}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \mathbf{v}_i\right)$$

The rate of change of the total momentum of all the particles

$$\boxed{\sum_{i=1}^n \sum_{j=1}^n \mathbf{F}_{ij}} = \sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \mathbf{v}_i\right)$$

I can rewrite the sum as

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\mathbf{F}_{ij}+\mathbf{F}_{ji}\right) = \sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \mathbf{v}_i\right)$$

Now using Newton's third law $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$

$$\boxed{0} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n}\left(m_i \mathbf{v}_i\right)$$

Integrating once

$$\mathbf{constant} = \sum_{i=1}^{n}\left(m_i \mathbf{v}_i\right)$$

where $\mathbf{constant}$ is a vector constant.

In particular the total momentum before and after a collision is the same.

Note that the proof required forces to be equal and opposite but not acting along the same line. (This will be required for conservation of *angular* momentum.)

4 Inelastic Collisions

Collisions can be elastic or inelastic.

Elastic collisions In an elastic collision the total energy is conserved as well as the total momentum. The collisions in Newton's cradle are approximately elastic.

Inelastic collisions In an inelastic collision the colliding particles coalesce into a single particle.

4.1 Two particles in one dimension

Two particles with masses $m_{1,2}$ and velocities $v_{1,2}$ collide and coalesce. What is the velocity, V, of the final particle? Initially the total momentum of the system is

$$p_{\text{initial}} = \boxed{m_1 v_1 + m_1 v_2}$$

After the collision and coalescence the system consists of a single particle of mass $m_1 + m_2$ travelling with velocity V. The final momentum of the system is given by

$$p_{\text{final}} = \boxed{(m_1 + m_2)\, V}$$

By conservation of momentum

$$p_{\text{final}} = \boxed{p_{\text{initial}}}$$

So

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2)\, V$$

and therefore

$$V = \boxed{\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}}$$

To statisticians the final velocity is the mass weighted average of the initial velocities of the particles.

Worked Example A particle of mass $1\,\text{kg}$ travelling at $2\,\text{m}\,\text{s}^{-1}$ collides with a particle of unknown mass travelling in the opposite direction at $4\,\text{m}\,\text{s}^{-1}$. The particles coalesce and the final assembly moves at a velocity of $1\,\text{m}\,\text{s}^{-1}$ in the direction the first particle was travelling in. What is the mass of the second particle?

Conservation of momentum

$$m_1 v_1 + m_2 v_2 = m_1 V + m_2 V$$

Solve for m_2

$$m_2 = \frac{m_1 (v_1 - V)}{V - v_2}$$

Khan Academy Conservation of Momentum.

Don't forget sign of v_2

$$m_2 = \frac{2 \times (2-1)}{1+4} = 0.4\,\mathrm{kg}$$

Worked Example In films people shot by handguns are often hurled backwards at high speeds. Is this reasonable?

Handguns have muzzle velocities ranging from 100 to $400\,\mathrm{m\,s^{-1}}$. An average Bond villain's henchman weighs $80\,$kg. Typical bullet weight $10^{-2}\,$kg. Final velocity of henchman:

$$v = \frac{10^{-2} \times 400}{80} = 0.05\,\mathrm{m\,s^{-1}}$$

or not very fast.

4.2 Two particles in higher dimensions

Two particles with masses $m_{1,2}$ and velocities $\mathbf{v}_{1,2}$ collide and coalesce. What is the (vector) velocity of the coalesced particle?

Initial momentum

$$\mathbf{p}_{\text{initial}} = \boxed{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}$$

Final momentum

$$\mathbf{p}_{\text{final}} = \boxed{(m_1 + m_2)\,\mathbf{V}}$$

Conservation of momentum

$$\mathbf{p}_{\text{initial}} = \boxed{\mathbf{p}_{\text{final}}}$$

And so

$$\mathbf{V} = \boxed{\frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}}.$$

Exam Question 2007 Three particles of masses $1\,\mathrm{kg}$, $3\,\mathrm{kg}$ and $2\,\mathrm{kg}$, simultaneously collide:

Before the collision, the middle particle was motionless, whereas the velocities of the other two were $1\,\mathrm{m\,s^{-1}}$ and $-3\,\mathrm{m\,s^{-1}}$ (see the diagram). Assuming that the particles collide non-elastically and coalesce, find their velocity after the collision. Which way will they be moving?

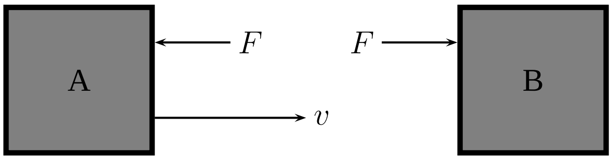
$$V = \frac{1\times 1 + 3\times 0 - 2\times 3}{1+3+2} = -\tfrac{5}{6}\,\mathrm{m\,s^{-1}}$$

–sign indicates they are travelling to the left.

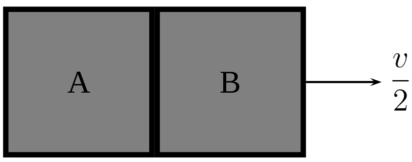
5 Forces during a collision

The law of conservation of momentum allows us to calculate the velocities of particles without worrying about the details of the forces. Sometimes it is useful to understand the details of those forces.

Consider two particles both of mass m, one is moving with velocity v, the other is stationary.



When they collide, the particles coalesce. By conservation of momentum, the velocity of the compound particle is $v/2$.



As the particles collide they exert forces on each other. Particle B exerts a force F on particle A and, by Newton's third law, particle A exerts an equal and opposite force, $-F$, on particle B.

The force is time dependent $F = F(t)$.

The force

$$-F(t) = m\frac{\mathrm{d}v_\mathrm{A}}{\mathrm{d}t} \qquad F(t) = m\frac{\mathrm{d}v_\mathrm{B}}{\mathrm{d}t}$$

Integrate with respect to time until both particles have the same velocity u and show that u must be the velocity predicted by conservation of momentum $u = v/2$

$$-\int_0^t F(t)\ \mathrm{d}t = m\int_v^u \mathrm{d}v_\mathrm{A} \qquad \int_0^t F(t)\ \mathrm{d}t = m\int_0^u \mathrm{d}v_\mathrm{B}$$

Evaluate the right hand side integrals

$$\int_0^t F(t)\ \mathrm{d}t = \boxed{m\,(v-u)} = \boxed{mu}$$

Solving the equation for u gives $u = v/2$. This result is independent of the form of $F(t)$; only the area under the graph of $F(t)$ is important.

6 The Sledgehammer Trick

In Jearl Walker's sledgehammer trick, *really don't try this at home*, there is an inelastic collision in which the sledge hammer, initially travelling downwards with speed v_0, is brought to rest.

The concrete block (while fracturing) exerts a force $F(t)$ on the sledgehammer. Newton's third law tells us that this force is exchanged between:

- The sledgehammer and the concrete block,
- The concrete block and the upper bed of nails,
- The upper bed of nails and Jearl Walker,
- Jearl Walker and the lower bed of nails,
- The lower bed of nails and the ground.

Ideally that force should remain below the value needed to drive the nails into any bones or important organs.

The value of the integral of $F(t)$ with respect to time is fixed. If the collision starts at $t = 0$ and ends at $t = T$

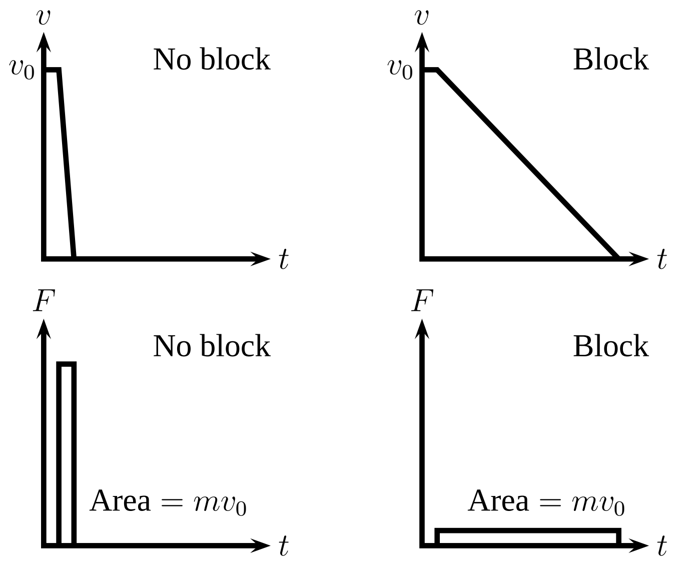
$$\int_0^T F(t)\,\mathrm{d}t = mv_0.$$

But, by increasing the time of the collision, the average value of the force.

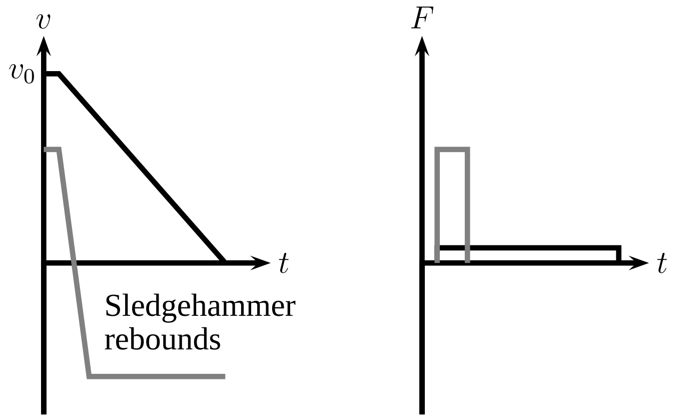
$$\bar{F} = \boxed{\frac{1}{T}\int_0^T F(t)\ \mathrm{d}t},$$

can be reduced

$$\bar{F}T = mv_0 \implies \bar{F} = \boxed{\frac{mv_0}{T}}.$$



The same idea is used in cars with crumple zones, collapsible steering columns and airbags.



A timid sledgehammer wielder who does not hit the concrete block hard enough to fracture it can be very dangerous