Tridiagonal Matrices: Thomas Algorithm

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The Thomas algorithm is an efficient way of solving tridiagonal matrix systems. It is based on LU decomposition in which the matrix system $Mx = r$ is rewritten as $LUx = r$ where L is a lower triangular matrix and U is an upper triangular matrix. The system can be efficiently solved by setting $Ux = \rho$ and then solving first $L\rho = r$ for ρ and then $Ux = \rho$ for x. The Thomas algorithm consists of two steps. In Step 1 decomposing the matrix into $M = LU$ *and* solving $L\rho = r$ are accomplished in a single downwards sweep, taking us straight from $Mx = r$ to $Ux = \rho$. In step 2 the equation $Ux = \rho$ is solved for x in an upwards sweep.

I. STAGE 1

In the first stage the matrix equation $Mx = r$ is converted to the form $Ux = \rho$. Initially the matrix equation looks like:

$$\begin{pmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 & 0 \\ 0 & 0 & 0 & a_5 & b_5 & c_5 \\ 0 & 0 & 0 & 0 & a_6 & b_6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix}$$

Row 1

$$b_1 x_1 + c_1 x_2 = r_1$$

Divide through by b_1

$$x_1 + \frac{c_1}{b_1} x_2 = \frac{r_1}{b_1}$$

Rewrite:

$$x_1 + \gamma_1 x_2 = \rho_1, \quad \gamma_1 = \frac{c_1}{b_1}, \quad \rho_1 = \frac{r_1}{b_1}$$

$$\begin{pmatrix} 1 & \gamma_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 & 0 \\ 0 & 0 & 0 & a_5 & b_5 & c_5 \\ 0 & 0 & 0 & 0 & a_6 & b_6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix}$$

Row 2.

$$a_2 x_1 + b_2 x_2 + c_2 x_3 = r_2$$

Use a_2 times row 1 of the matrix to eliminate the first term

$$a_2 \left(x_1 + \gamma_1 x_2 = \rho_1 \right)$$

$$\begin{array}{r|llll} \text{Row 2} & a_2 x_1 + & b_2 x_2 + c_2 x_3 = & r_2 \\ a_2 \times \text{Row 1} & a_2 x_1 + & a_2 \gamma_1 x_2 \qquad = & a_2 \rho_1 \\ \hline \text{New Row 2} & & (b_2 - a_2\gamma_1)\, x_2 + c_2 x_3 = & r_2 - a_2 \rho_1 \end{array}$$

Divide through by $(b_2 - a_2\gamma_1)$ to get

$$x_1 + \frac{c_2}{b_2 - a_2\gamma_1} x_2 = \frac{r_2 - a_2\rho_1}{b_2 - a_2\gamma_1}$$

We can rewrite this as

$$x_2 + \gamma_2 x_3 = \rho_2, \quad \gamma_2 = \frac{c_2}{b_2 - a_2\gamma_1}, \quad \rho_2 = \frac{r_2 - a_2\rho_1}{b_2 - a_2\gamma_1}.$$

$$\begin{pmatrix} 1 & \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 & 0 \\ 0 & 0 & 0 & a_5 & b_5 & c_5 \\ 0 & 0 & 0 & 0 & a_6 & b_6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix}$$

Row 3.

$$a_3 x_2 + b_3 x_3 + c_3 x_4 = r_3$$

Use a_3 times row 2 of the matrix to eliminate the first term

$$a_3 \left(x_2 + \gamma_2 x_3 = \rho_2 \right)$$

$$\begin{array}{r|llll} \text{Row 3} & a_3 x_2 + & b_3 x_3 + c_3 x_4 = & r_3 \\ a_3 \times \text{Row 2} & a_3 x_2 + & a_3 \gamma_2 x_3 \qquad = & a_3 \rho_2 \\ \hline \text{New Row 3} & & (b_3 - a_3\gamma_2)\, x_3 + c_3 x_4 = & r_3 - a_3 \rho_2 \end{array}$$

Divide through by $(b_3 - a_3\gamma_2)$ to get

$$x_2 + \frac{c_3}{b_3 - a_3\gamma_2} x_3 = \frac{r_3 - a_3\rho_2}{b_3 - a_3\gamma_2}$$

We can rewrite this as

$$x_3 + \gamma_3 x_4 = \rho_3, \quad \gamma_3 = \frac{c_3}{b_3 - a_3\gamma_2}, \quad \rho_3 = \frac{r_3 - a_3\rho_2}{b_3 - a_3\gamma_2}.$$

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$$\begin{pmatrix} 1 & \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & \gamma_3 & 0 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 & 0 \\ 0 & 0 & 0 & a_5 & b_5 & c_5 \\ 0 & 0 & 0 & 0 & a_6 & b_6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix}$$

Row 4.

$$a_4 x_3 + b_4 x_4 + c_4 x_5 = r_4$$

Use a_4 times row 3 of the matrix to eliminate the first term

$$a_4 \left(x_3 + \gamma_3 x_4 = \rho_3 \right)$$

$$\begin{array}{r|lllll} \text{Row 4} & a_4 x_3 + & & b_4 x_4 + c_4 x_5 & = r_4 \\ a_4 \times \text{Row 3} & a_4 x_3 + & & a_4 \gamma_3 x_4 & = a_4 \rho_3 \\ \hline \text{New Row 4} & & & (b_4 - a_4 \gamma_3) x_4 + c_4 x_5 & = r_4 - a_4 \rho_3 \end{array}$$

Divide through by $(b_4 - a_4 \gamma_3)$ to get

$$x_3 + \frac{c_4}{b_4 - a_4 \gamma_3} x_4 = \frac{r_4 - a_4 \rho_3}{b_4 - a_4 \gamma_3}$$

We can rewrite this as

$$x_4 + \gamma_4 x_5 = \rho_4, \quad \gamma_4 = \frac{c_4}{b_4 - a_4 \gamma_3}, \quad \rho_4 = \frac{r_4 - a_4 \rho_3}{b_4 - a_4 \gamma_3}.$$

$$\begin{pmatrix} 1 & \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & \gamma_4 & 0 \\ 0 & 0 & 0 & a_5 & b_5 & c_5 \\ 0 & 0 & 0 & 0 & a_6 & b_6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ r_5 \\ r_6 \end{pmatrix}$$

Row 5.

$$a_5 x_4 + b_5 x_5 + c_5 x_6 = r_5$$

Use a_5 times row 4 of the matrix to eliminate the first term

$$a_5 \left(x_4 + \gamma_4 x_5 = \rho_4 \right)$$

$$\begin{array}{r|lllll} \text{Row 5} & a_5 x_4 + & & b_5 x_5 + c_5 x_6 & = r_5 \\ a_5 \times \text{Row 4} & a_5 x_4 + & & a_5 \gamma_4 x_5 & = a_5 \rho_4 \\ \hline \text{New Row 5} & & & (b_5 - a_5 \gamma_4) x_5 + c_5 x_6 & = r_5 - a_5 \rho_4 \end{array}$$

Divide through by $(b_5 - a_5 \gamma_4)$ to get

$$x_4 + \frac{c_5}{b_5 - a_5 \gamma_4} x_5 = \frac{r_5 - a_5 \rho_4}{b_5 - a_5 \gamma_4}$$

We can rewrite this as

$$x_5 + \gamma_5 x_6 = \rho_5, \quad \gamma_5 = \frac{c_5}{b_5 - a_5 \gamma_4}, \quad \rho_5 = \frac{r_5 - a_5 \rho_4}{b_5 - a_5 \gamma_4}.$$

$$\begin{pmatrix} 1 & \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & \gamma_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & \gamma_5 \\ 0 & 0 & 0 & 0 & a_6 & b_6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ r_6 \end{pmatrix}$$

Row 6

$$a_6 x_5 + b_6 x_6 = r_6$$

Use a_6 times row 5 to eliminate the first term.

$$a_6 x_5 + a_6 \gamma_5 x_6 = a_6 \rho_5$$

Resulting in

$$(b_6 - a_6 \gamma_5) x_6 = r_6 - a_6 \rho_5$$

Divide through by $b_6 - a_6 \gamma_5$ to get

$$x_6 = \rho_6, \quad \rho_6 = \frac{r_6 - a_6 \rho_5}{b_6 - a_6 \gamma_5}$$

$$\begin{pmatrix} 1 & \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & \gamma_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & \gamma_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \end{pmatrix}$$

At this point the matrix has been reduced to upper diagonal form, so our equations are in the form $Ux = \rho$.

II. STAGE 2

The matrix equation is now in a form which is trivial to solve for x. We start with the last row and work our way up. The final equation is already solved

$$x_6 = \rho_6.$$

$$\begin{pmatrix} 1 & \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & \gamma_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & \gamma_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \end{pmatrix}$$

Row 5: $x_5 + \gamma_5 x_6 = \rho_5$.
Rearrange to get: $x_5 = \rho_5 - \gamma_5 x_6$.

$$\begin{pmatrix} 1 & \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & \gamma_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & \gamma_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \end{pmatrix}$$

Row 4: $x_4 + \gamma_4 x_5 = \rho_4$.
Rearrange to get: $x_4 = \rho_4 - \gamma_4 x_5$.

$$\begin{pmatrix} 1 & \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & \gamma_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & \gamma_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \end{pmatrix}$$

Row 3: $x_3 + \gamma_3 x_4 = \rho_3$.
Rearrange to get: $x_3 = \rho_3 - \gamma_3 x_4$.

$$\begin{pmatrix} 1 & \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & \gamma_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & \gamma_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \end{pmatrix}$$

Row 2: $x_2 + \gamma_2 x_3 = \rho_2$.
Rearrange to get: $x_2 = \rho_2 - \gamma_2 x_3$.

$$\begin{pmatrix} 1 & \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & \gamma_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & \gamma_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \end{pmatrix}$$

Row 1: $x_1 + \gamma_1 x_2 = \rho_1$.
Rearrange to get: $x_1 = \rho_1 - \gamma_1 x_2$.

$$\begin{pmatrix} 1 & \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & \gamma_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & \gamma_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \end{pmatrix}$$

At this point x, the solution to the matrix equation, is fully determined.

III. IN PRACTICE

The Thomas algorithm is used because it is fast and because tridiagonal matrices often occur in practice. (This argument is slightly circular because people often manipulate the problems they are working on to reduce them to solving a tridiagonal matrix problem.) Although it is rare, the algorithm can be unstable if $b_i - a_i\gamma_{i-1}$ is zero or numerically zero for any i. This will occur if the tridiagonal matrix is singular, but in rare cases can occur if it is non-singular. The condition for the algorithm to be stable is

$$\|b_i\| > \|a_i\| + \|c_i\|$$

for all i. The matrix problems which result from the discretisation of partial differential equations nearly all satisfy this criterion.

If the algorithm is numerically unstable then you must rearrange the equations: known as pivoting. Standard LU decomposition algorithms for full or banded matrices include pivoting. (But first you should check to make sure you have not made a mistake in formulating the problem.)