UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF MATHEMATICS & STATISTICS

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS 4414	SEMESTER: Repeat 2009
MODULE TITLE: Theoretical Mechanics	DURATION OF EXAM: $2\frac{1}{2}$hrs
LECTURER: Dr. S. Soussi	GRADING SCHEME: 80%

INSTRUCTIONS TO CANDIDATES:
The exam in divided into 3 Sections A, B, and C.
Answer all questions in Section A.
Answer all questions in Section B or all questions in Section C.
Say clearly which of sections B or C you will answer.

Section A

Question 1 (10%)

Consider a closed system of two particles of masses m_1 and m_2 located at $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. We suppose that the particles are moving (i.e. $\mathbf{r}_1$ and $\mathbf{r}_2$ depend on time t).

(a) Write the momentum $\mathbf{P}$ of the system in terms of m_1, m_2, $\dot{\mathbf{r}}_1$, and $\dot{\mathbf{r}}_2$. [2%]

(b) Show that $\mathbf{P}$ does not depend on time. [3%]

(c) Write the angular momentum $\mathbf{A}_M$ of the system with respect to a fixed point M located at $\mathbf{r}_M$ of the system in terms of m_1, m_2, $\mathbf{r}_M$, $\mathbf{r}_1$, $\mathbf{r}_2$, $\dot{\mathbf{r}}_1$, and $\dot{\mathbf{r}}_2$. [2%]

(d) Supposing that the inter-particle interactions are parallel to the straight line going through the particles, show that $\mathbf{A}_M$ does not depend on time. [3%]

Question 2 (15%)

Two identical particles of masses m collide. Their velocities before the collision $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal as shown in figure 1.

Assuming that the system is closed and that the particles collide non-elastically and coalesce:

(a) find their velocity $\mathbf{v}$ after the collision, in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. [7%]

(b) find $\cos\alpha$ in terms of $\mathbf{v}_1$, and $\mathbf{v}_2$, where α is the angle between $\mathbf{v}_1$ and $\mathbf{v}$. [8%]

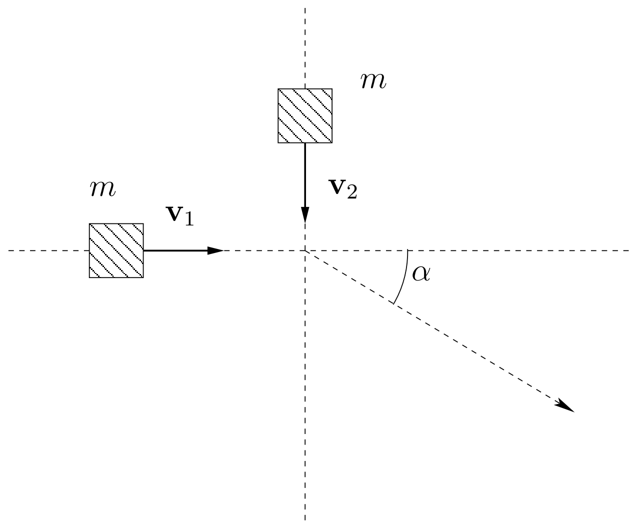

Figure 1: Collision

Question 3 (20%)

Two identical particles of mass m are attached to three identical springs (modulus k and unperturbed length L) as shown of figure 2. The left and right springs are attached to fixed walls. The distance between the fixed walls is denoted by D.

We suppose that the only foces affecting the particles are the spring tensions. We do not consider the gravity force!

We denote by x_i and $p_i = m_i \dot{x}_i$ the position and momentum of particle i ($i = 1, 2$), respectively.

(a) Write down the expression for the total energy H of the system of this system in terms of x_i and p_i ($i = 1, 2$). Note that the potential energy is equal to the potential energy of the 3 springs. 10%

(b) Write down the 4 Hamiltonian equations of this system. 10%

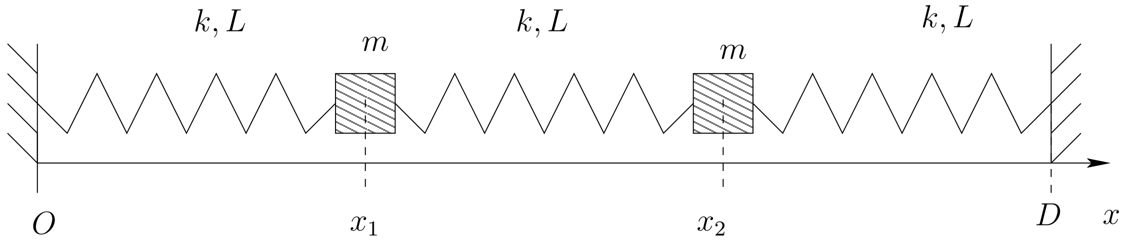

Figure 2: 2 particle system

Section B

Question 1 (20%)

A particle of mass m is attached to the bottom of a vertical spring. The top of the spring is attached to the ceiling. The spring has modulus k and unperturbed length L.

We suppose the particle is subject to the gravity force and the spring tension.

(a) Using the Newton's first law, find the equilibrium position x_e in the coordinate system represented on figure 3. [4%]

(b) Find the potential energy U of the particle when it has position x. (The potential energy is the sum of the gravitational potential energy and of the spring potential energy) [4%]

(c) Find the position x_0 such that the potential energy U is minimum. [4%]

(d) What do you notice? Is that surprising? [4%]

(e) The particle starts moving from the equilibrium position with initial velocity $\mathbf{v_0} = v_0\mathbf{e}_x$, where $\mathbf{e}_x$ is the unit vector oriented downwards. Write down the total energy of the particle E in terms of the position x, the velocity $\dot{x}$, and of the problem constants. [4%]

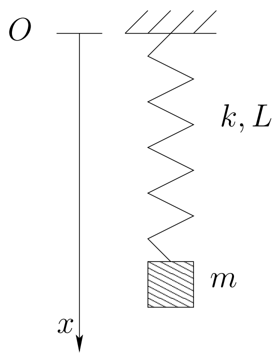

Figure 3: Spring

Question 2 (15%)

Consider the following dynamical system.

$$\begin{cases} \dot{\phi} = (\phi+1)(\psi-1), \\ \dot{\psi} = (\phi-2)(\psi+1). \end{cases}$$

(a) Show that the fixed points of this dynamical system are exactly $(-1,-1)$ and $(2,-1)$. [5%]

(b) Show that one of these fixed points is stable and the other is unstable. [10%]

Section C

Question 1 (15%)

A trolley is going up a slope making an angle θ with the horizontal plane. We neglect any friction force and suppose the trolley exclusively subject to the gravity force $m\mathbf{g}$ and to the normal reaction of the plane $\mathbf{R}_N$.

Supposing the motion starts at the origin O with an initial velocity $\mathbf{v}_0 = v_0\mathbf{x}$ ($v_0 > 0$), find the time t_f when it will be back to its initial position in terms of g and θ.

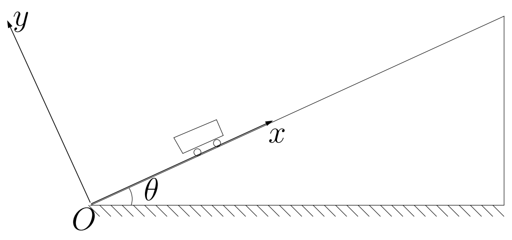

Figure 4: Trolley on a slope

Question 2 (20%)

A satellite of mass m is rotating about a planet of mass M and radius R with angular velocity ω. The satellite is at height H. We suppose that the satellite is subject only to the gravitational attraction of the planet.

(a) Find the acceleration $\mathbf{a}$ of the satellite in terms of R, H, ω, and the unit vector $\mathbf{e}_r$ oriented from the center of the planet to the satellite. (see figure 5) [10%]

(b) By applying Newton's second law and Newton's law of gravity, express the angular velocity ω in terms of M, R, H, and the gravitational constant $\mathcal{G}$. [10%]

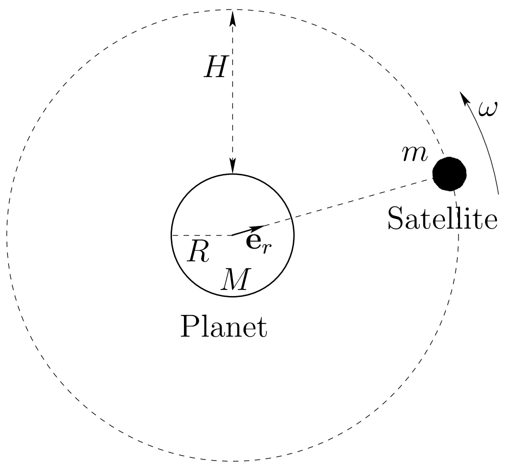

Figure 5: Satellite

Theoretical mechanics : summary

Kinematics

1. Position vector $\mathbf{r}$, velocity $\mathbf{v}$, and acceleration of a particle are related by:

$$\mathbf{v} = \dot{\mathbf{r}},$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}.$$

2. 1D motion with constant velocity v:

$$x = vt + x_0$$

3. 1D motion with a constant acceleration a:

$$v = at + v_0,$$

$$x = \frac{a}{2}t^2 + v_0 t + x_0,$$

where x_0 and v_0 are the position and velocity at $t = 0$, respectively.

4. Rotation with constant angular velocity ω (frequency $\nu = \frac{\omega}{2\pi}$) along a circle of radius R:

   - polar coordinates
   $$\begin{cases} r &= R, \\ \theta &= \omega t + \theta_0, \end{cases}$$
   - Cartesian coordinates:
   $$\begin{cases} x &= R\cos(\omega t + \theta_0), \\ y &= R\sin(\omega t + \theta_0), \end{cases}$$
   - linear velocity:
   $$v = R\omega,$$
   - acceleration:
   $$a = R\omega^2,$$

   where θ_0 is the value of θ at $t = 0$.

5. Rotation with constant angular acceleration α:

$$\omega = \alpha t + \omega_0,$$

$$\theta = \frac{\alpha}{2}t^2 + \omega_0 t + \theta_0,$$

where ω is the angular velocity, θ is the angular coordinate, ω_0 is the angular velocity at $t = 0$, and θ_0 is the angular coordinate at $t = 0$.

Dynamics

1. Newton's Second Law:
$$m\mathbf{a} = \mathbf{F}.$$

2. For a sliding body, the friction force is $F_{\text{rf}} = kN$, where N is the normal reaction force. It is oriented in the opposite sense of the motion.

3. Conserved quantities:

   - linear momentum
   $$\mathbf{P} = \sum_i m_i \mathbf{v}_i,$$
   - angular momentum with respect to the origin
   $$\mathbf{A}_O = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i,$$
   - angular momentum with respect to an arbitrary point P
   $$\mathbf{A}_P = \sum_i m_i (\mathbf{r}_i - \mathbf{r}_P) \times \dot{\mathbf{r}}_i,$$
   - total energy
   $$E = U(x_1, x_2, \ldots) + \sum_i \frac{m_i v_i^2}{2},$$
   where U is the potential energy.

4. A conservative force $\mathbf{F}$ and the corresponding potential energy U are related by
$$\mathbf{F} = \nabla U.$$

5. The potential energy and force for a spring of modulus k and unperturbed length L are
$$U = \frac{k(L' - L)^2}{2},$$
$$F = k|L' - L|,$$
where L' is the current length of the spring. The direction of $\mathbf{F}$ is such that it tries to bring the spring back to its unperturbed configuration.

6. The potential energy U and force F for a particle of mass m located at a height H, in the Earth's gravitational field are

   - locally:
   $$U = mgH, \qquad \mathbf{F} = m\mathbf{g},$$

- globally

$$U = -\frac{GM_E m}{R_E + H}, \qquad \mathbf{F} = -\frac{GM_E m}{(R_E+H)^2}\frac{\mathbf{OP}}{OP}.$$

where G is the gravitational constant, M_E is the Earth's mass, R_E is the Earth's radius, O is the Earth's center, and P is the particle position.

$$G \approx 6.7\times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}, \qquad M_E \approx 6.0\times 10^{24}\text{kg}, \qquad R_E \approx 6.4\times 10^{6}\text{m}.$$

7. The angular velocity of a body rotating along a circular orbit around a much heavier body of mass M is

$$\omega = \sqrt{\frac{GM}{R^3}}$$

where R is the orbit radius.

Oscillations

The equation of forced linear pendulum with small amplitude is

$$\ddot{\phi} + 2c\dot{\phi} + \omega^2\phi = F_0\cos(\Omega t),$$

where c is the friction coefficient, $\omega^2 = \frac{L}{g}$ is the natural frequency of the pendulum, L is the length of the pendulum, F_0 and Ω_0 are the amplitude and frequency of the external forcing.

Hamiltonian mechanics

1. The Hamiltonian equations are:

$$\begin{cases} \dot{x}_j = & \frac{\partial H}{\partial p_j}, \\ \dot{p}_j = & -\frac{\partial H}{\partial x_j}, \end{cases} \qquad 1 \le j \le n$$

2. The Poisson brackets of functions $F(x_1,\ldots x_n,p_1,\ldots,p_n)$ and $G(x_1,\ldots x_n,p_1,\ldots,p_n)$ are

$$\{F,G\} = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial x_j} - \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial p_j}\right).$$

3. A transformation

$$x_i' = x_i'(x_1,\ldots,x_n,p_1,\ldots,p_n), \qquad p_i' = p_i'(x_1,\ldots,x_n,p_1,\ldots,p_n),$$

is canonical if and only if

$$\{x_i',p_k'\} = \begin{cases} -1 & \text{if } i=k, \\ 0 & \text{if } i\neq k, \end{cases}$$

$$\{x_i',x_k'\} = 0,$$

$$\{p_i',p_k'\} = 0.$$

4. The Lagrangian equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j}0, \qquad 1 \le j \le n.$$

Stability of dynamical systems

Let x_F be a fixed point of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} f_1(x_1, \ldots, x_k) \\ f_2(x_1, \ldots, x_k) \\ \vdots \\ f_k(x_1, \ldots, x_k) \end{bmatrix}.$$

Then,

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_k} \end{bmatrix},$$

is the Jacobian matrix of the system at $\mathbf{x}_F$, with λ_1, λ_2,..., λ_k being its eigenvalues.

- If $\mathrm{Re}(\lambda_j) < 0$ for all j then $\mathbf{x}_F$ is asymptotically stable.
- If $\mathrm{Re}(\lambda_j) > 0$ for some j then $\mathbf{x}_F$ is unstable.
- If $\mathrm{Re}(\lambda_j) < 0$ for some j, and $\mathrm{Re}(\lambda_j) = 0$ for the remaining j, then the test is inconclusive.