UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF MATHEMATICS & STATISTICS

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS 4414

SEMESTER: Spring 2009

MODULE TITLE: Theoretical Mechanics

DURATION OF EXAM: $2\frac{1}{2}$hrs

LECTURER: Dr. S. Soussi

GRADING SCHEME: 80%

EXTERNAL EXAMINER: Prof. J. Flavin

INSTRUCTIONS TO CANDIDATES:
The exam in divided into 3 Sections A, B, and C.
Answer all questions in Section A.
Answer all questions in Section B or all questions in Section C.
Say clearly which of sections B or C you will answer.

Section A

Question 1 (10%)

Consider a closed system of two particles of masses m_1 and m_2 located at $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. We suppose that the particles are moving (i.e. $\mathbf{r}_1$ and $\mathbf{r}_2$ depend on time t).

(a) Write the momentum $\mathbf{P}$ of the system in terms of m_1, m_2, $\dot{\mathbf{r}}_1$, and $\dot{\mathbf{r}}_2$. [2%]

(b) Show that $\mathbf{P}$ does not depend on time. [3%]

(c) Write the angular momentum $\mathbf{A}_M$ of the system with respect to a fixed point M located at $\mathbf{r}_M$ of the system in terms of m_1, m_2, $\mathbf{r}_M$, $\mathbf{r}_1$, $\mathbf{r}_2$, $\dot{\mathbf{r}}_1$, and $\dot{\mathbf{r}}_2$. [2%]

(d) Supposing that the inter-particle interactions are parallel to the straight line going through the particles, show that $\mathbf{A}$ does not depend on time. [3%]

Question 2 (15%)

Two particles of masses m_1 and $m_2 = 2m_1$ collide. Their velocities before the collision $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal as shown in figure 1.

Assuming that the system is closed and that the particles collide non-elastically and coalesce:

(a) find their velocity $\mathbf{v}$ after the collision, in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. [7%]

(b) find $\cos\alpha$ in terms of $\mathbf{v}_1$, and $\mathbf{v}_2$, where α is the angle between $\mathbf{v}_1$ and $\mathbf{v}$. [8%]

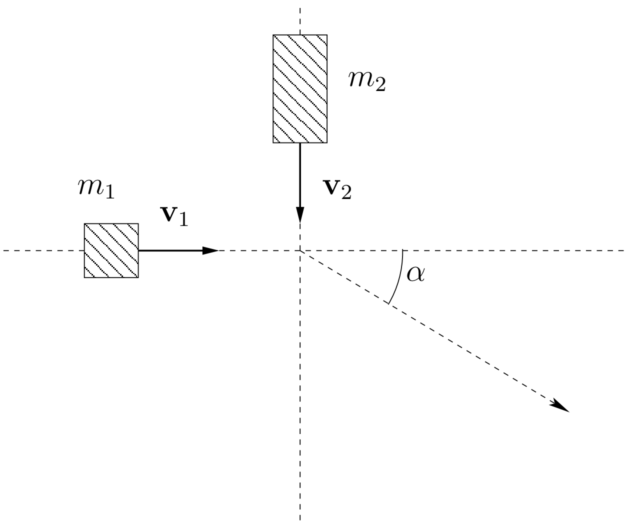

Figure 1: Collision

Question 3 (20%)

Two identical particles of mass m are attached to three identical springs (modulus k and unperturbed length L) as shown of figure 2. The top and bottom springs are attached to fixed supports. The distance between the fixed supports is denoted by D.

We denote by x_i and $p_i = m_i \dot{x}_i$ the position and momentum of particle i ($i = 1, 2$), respectively.

(a) Write down the expression for the total energy H of the system of this system in terms of x_i and p_i ($i = 1, 2$). Note that the potential energy is equal to the sum of the gravitational potential energy of the two particles and of the potential energy of the 3 springs. 10%

(b) Write down the 4 Hamiltonian equations of this system. 10%

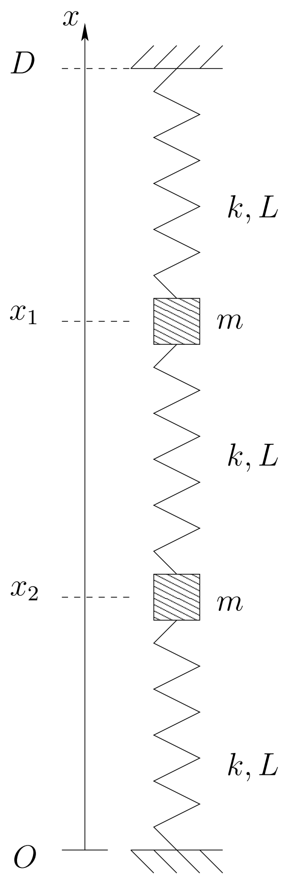

Figure 2: 2 particle system

Section B

Question 1 (20%)

A body of mass m is attached to the top of a vertical spring, the bottom of the spring is attached to the ground. The spring has modulus k and unperturbed length L. The position of the body on the x-axis is denoted by $x(t)$.

The body can move between two vertical walls and is subject to a friction force given by:

$$\mathbf{F}_{\text{fr}} = -\alpha \dot{x} \mathbf{e}_x,$$

where $\mathbf{e}_x$ is the unit vector of the x-axis.

(a) Find the equilibrium position x_0 of the body. 5%

(b) Write down the equation of motion of the body. 5%

(c) Prove that $m = \frac{\alpha^2}{4k}$ corresponds to the case of critical damping. 4%

(d) In the case of critical damping, express the position x as a function of time, supposing that the body starts its motion at $t = 0$ from the equilibrium position x_0 with velocity v_0. 6%

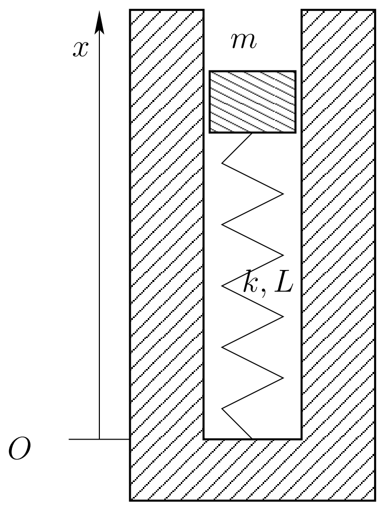

Figure 3: Shock absorber

Question 2 (15%)

Consider the following dynamical system.

$$\begin{cases} \dot{\phi} &= (\phi + 1)\psi, \\ \dot{\psi} &= \phi(\psi - 1). \end{cases}$$

(a) Show that the fixed points of this dynamical system are exactly $(0, 0)$ and $(-1, 1)$. 5%

(b) Examine the stability of each fixed point. 10%

Section C

Question 1 (15%)

A satellite of mass m is rotating about a planet of mass M and radius R with angular velocity ω. The satellite is at height H. We suppose that the satellite is subject only to the gravitational attraction of the planet.

(a) Find the acceleration $\mathbf{a}$ of the satellite in terms of R, H, ω, and the unit vector $\mathbf{e}_r$ oriented from the center of the planet to the satellite. (see figure 4) 6%

(b) By applying Newton's second law and Newton's law of gravity, express the angular velocity ω in terms of M, R, H, and the gravitational constant $\mathcal{G}$. 9%

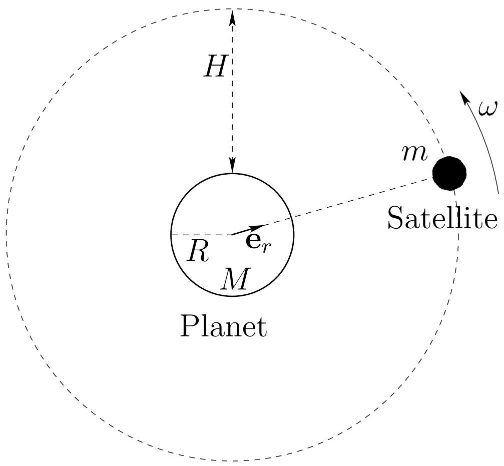

Figure 4: Satellite

Question 2 (20%)

Two spherical bodies of radii R_1 and R_2 and masses m_1 and m_2 are attracted to each other through gravity. The initial velocities of the bodies are zero, the initial distance separating them is infinitely large. Find their velocities when they collide.

Theoretical mechanics : summary

Kinematics

1. Position vector $\mathbf{r}$, velocity $\mathbf{v}$, and acceleration of a particle are related by:

$$\mathbf{v} = \dot{\mathbf{r}},$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}.$$

2. 1D motion with constant velocity v:

$$x = vt + x_0$$

3. 1D motion with a constant acceleration a:

$$v = at + v_0,$$

$$x = \frac{a}{2}t^2 + v_0 t + x_0,$$

where x_0 and v_0 are the position and velocity at $t = 0$, respectively.

4. Rotation with constant angular velocity ω (frequency $\nu = \frac{\omega}{2\pi}$) along a circle of radius R:

- polar coordinates

$$\begin{cases} r &= R, \\ \theta &= \omega t + \theta_0, \end{cases}$$

- Cartesian coordinates:

$$\begin{cases} x &= R\cos(\omega t + \theta_0), \\ y &= R\sin(\omega t + \theta_0), \end{cases}$$

- linear velocity:

$$v = R\omega,$$

- acceleration:

$$a = R\omega^2,$$

where θ_0 is the value of θ at $t = 0$.

5. Rotation with constant angular acceleration α:

$$\omega = \alpha t + \omega_0,$$

$$\theta = \frac{\alpha}{2}t^2 + \omega_0 t + \theta_0,$$

where ω is the angular velocity, θ is the angular coordinate, ω_0 is the angular velocity at $t = 0$, and θ_0 is the angular coordinate at $t = 0$.

Dynamics

1. Newton's Second Law:
$$m\mathbf{a} = \mathbf{F}.$$

2. For a sliding body, the friction force is $F_{\text{rf}} = kN$, where N is the normal reaction force. It is oriented in the opposite sense of the motion.

3. Conserved quantities:

   - linear momentum
   $$\mathbf{P} = \sum_i m_i \mathbf{v}_i,$$
   - angular momentum with respect to the origin
   $$\mathbf{A}_O = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i,$$
   - angular momentum with respect to an arbitrary point P
   $$\mathbf{A}_P = \sum_i m_i (\mathbf{r}_i - \mathbf{r}_P) \times \dot{\mathbf{r}}_i,$$
   - total energy
   $$E = U(x_1, x_2, \ldots) + \sum_i \frac{m_i v_i^2}{2},$$
   where U is the potential energy.

4. A conservative force $\mathbf{F}$ and the corresponding potential energy U are related by
$$\mathbf{F} = \nabla U.$$

5. The potential energy and force for a spring of modulus k and unperturbed length L are
$$U = \frac{k(L' - L)^2}{2},$$
$$F = k|L' - L|,$$
where L' is the current length of the spring. The direction of $\mathbf{F}$ is such that it tries to bring the spring back to its unperturbed configuration.

6. The potential energy U and force F for a particle of mass m located at a height H, in the Earth's gravitational field are

   - locally:
   $$U = -mgH, \qquad \mathbf{F} = m\mathbf{g},$$

- globally

$$U = -\frac{\mathrm{G}M_E m}{R_E + H}, \qquad \mathbf{F} = -\frac{\mathrm{G}M_E m}{(R_E+H)^2}\frac{\mathbf{OP}}{OP}.$$

  where G is the gravitational constant, M_E is the Earth's mass, R_E is the Earth's radius, O is the Earth's center, and P is the particle position.

$$\mathrm{G} \approx 6.7 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}, \qquad M_E \approx 6.0 \times 10^{24}\mathrm{kg}, \qquad R_E \approx 6.4 \times 10^6\mathrm{m}.$$

7. The angular velocity of a body rotating along a circular orbit around a much heavier body of mass M is

$$\omega = \sqrt{\frac{\mathrm{G}M}{R^3}}$$

   where R is the orbit radius.

Oscillations

The equation of forced linear pendulum with small amplitude is

$$\ddot{\phi} + 2c\dot{\phi} + \omega^2\phi = F_0\cos(\Omega t),$$

where c is the friction coefficient, $\omega^2 = \frac{L}{g}$ is the natural frequency of the pendulum, L is the length of the pendulum, F_0 and Ω_0 are the amplitude and frequency of the external forcing.

Hamiltonian mechanics

1. The Hamiltonian equations are:

$$\begin{cases} \dot{x}_j = & \frac{\partial H}{\partial p_j}, \\ \dot{p}_j = & -\frac{\partial H}{\partial x_j}, \end{cases} \qquad 1 \le j \le n$$

2. The Poisson brackets of functions $F(x_1,\dots x_n,p_1,\dots,p_n)$ and $G(x_1,\dots x_n,p_1,\dots,p_n)$ are

$$\{F,G\} = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial x_j} - \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial p_j}\right).$$

3. A transformation

$$x_i' = x_i'(x_1,\dots,x_n,p_1,\dots,p_n), \qquad p_i' = p_i'(x_1,\dots,x_n,p_1,\dots,p_n),$$

   is canonical if and only if

$$\{x_i',p_k'\} = \begin{cases} -1 & \text{if } i = k, \\ 0 & \text{if } i \ne k, \end{cases}$$
$$\{x_i',x_k'\} = 0,$$
$$\{p_i',p_k'\} = 0.$$

4. The Lagrangian equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j}0, \qquad 1 \le j \le n.$$

Stability of dynamical systems

Let x_F be a fixed point of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} f_1(x_1, \dots, x_k) \\ f_2(x_1, \dots, x_k) \\ \vdots \\ f_k(x_1, \dots, x_k) \end{bmatrix}.$$

Then,

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_k} \end{bmatrix},$$

is the Jacobian matrix of the system at $\mathbf{x}_F$, with λ_1, λ_2,..., λ_k being its eigenvalues.

- If $\text{Re}(\lambda_j) < 0$ for all j then $\mathbf{x}_F$ is asymptotically stable.
- If $\text{Re}(\lambda_j) > 0$ for some j then $\mathbf{x}_F$ is unstable.
- If $\text{Re}(\lambda_j) < 0$ for some j, and $\text{Re}(\lambda_j) = 0$ for the remaining j, then the test is inconclusive.