UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

COLLEGE OF INFORMATICS AND ELECTRONICS

DEPARTMENT OF MATHEMATICS & STATISTICS

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: **MS4414** **SEMESTER:** **Annual Repeats 07-08**

MODULE TITLE: **Theoretical Mechanics** **DURATION OF EXAM:** **2 hours**

LECTURER: **Prof Eugene Benilov** **GRADING SCHEME:** **Examination 100%**

EXTERNAL EXAMINER: **Prof J King**

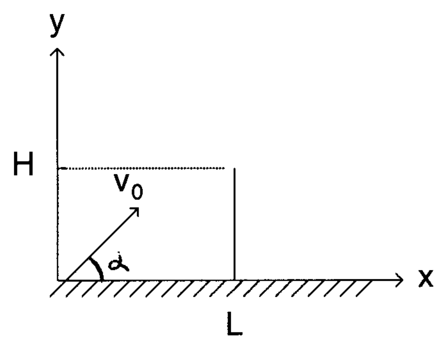
INSTRUCTIONS TO CANDIDATES

Please attempt all questions

MS4414 **Theoretical Mechanics** **Repeat 2007/08** **Prof. E. Benilov**

Question 1 [25 marks]

1) A stone is projected (under gravity, with air friction neglected) with velocity v_0, at an angle α towards a "wall" of height H located at a distance L:



Find for which values of α the stone goes over the wall.

2) The trajectory of a particle is given by

$$x = t \cos \tfrac{1}{2} t, \qquad y = \sin \tfrac{1}{2} t \qquad t: 0 \to \tfrac{3\pi}{2}\ .$$

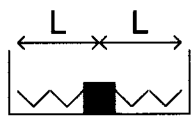
Sketch the trajectory of the particle.

3) The same as in part 2, but for a particle with polar coordinates, given by

$$r = \frac{1}{\cos t}, \quad \theta = t, \quad t: 0 \to \pi/2.$$

Question 2 [25 marks]

1) A particle of mass m is attached to two identical springs of module k and unperturbed length L, which are in turn attached to walls:



Initially, the particle's velocity is v. Find how far it will get.

2) Find the velocity of a particle of mass m after it has slid down a wedge with an angle α and height H. How long has it taken for the particle to have reached the "foot" of the wedge?

3) The same as in the previous problem, but including the force of friction (the magnitude of F_{fr} is given by $|F_{fr}| = \mu\,|N|$, where N is the reaction force and μ is the friction coefficient).

4) The drag force affecting a skydiver, of mass m, with his parachute closed is $F_{dr} = -\mu v$ (v is the velocity of the fall, μ is the friction coefficient). Find the so-called "terminal velocity", i.e. the velocity for which the gravitational and drag forces are exactly balanced.

Question 3 [25 marks]

1) Consider a one-dimensional system which consists of two particles of masses m_1 and m_2, with coordinates x_1 and x_2 ($x_1 < x_2$) interacting through gravity. Write down the expression for the Lagrangian of the system, and derive the Lagrangian form of the governing equations.

2) Consider the two-dimensional equivalent of the system described in Question 1 above, i.e. two particles of masses m_1 and m_2, with coordinates (x_1,y_1) and (x_2,y_2), interacting through gravity.

Write down the expression for the kinetic and potential energies of the system.

3) Find and examine the fixed points of

$$\dot{\phi} = \phi + \psi^2 + \psi, \qquad \dot{\psi} = \phi + \psi^2 + 1.$$

Question 4 [25 marks]

Consider

$$\ddot{\phi} + 2c\,\dot{\phi} + (1 + \varepsilon \cos 2\Omega t)\,\phi = 0, \qquad (1)$$

where ε, $c \ll 1$ and $\Omega \approx 1$.

1) Seek a solution in the form

$$\phi = B(t) \cos \Omega t + D(t) \sin \Omega t. \qquad (2)$$

2) Upon substitution of (2) into (1), omit small terms involving $\ddot{B}$, $\ddot{D}$, $c\dot{B}$, and $c\dot{D}$.

3) Omit the non-resonant terms, i.e. terms involving $\cos 3\Omega t$ and $\sin 3\Omega t$.

4) Collect like terms and obtain a set of equations for $B(t)$ and $D(t)$. Reduce this set to a single equation for $D(t)$ and find its general solution.

5) On the basis of this solution, determine the range of Ω for which parametric resonance occurs in the system.

THEORETICAL MECHANICS (SUMMARY)

Kinematics

1) Position vector **r**, velocity **v**, and acceleration **a** of a particle:

$$\mathbf{v} = \dot{\mathbf{r}}, \qquad \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}.$$

2) 1D motion with constant velocity v:

$$x = x_0 + v\,t.$$

3) 1D motion with constant acceleration a:

$$v = v_0 + a\,t, \qquad x = x_0 + v_0\,t + \frac{a\,t^2}{2}$$

(x_0 and v_0 are the initial coordinate and velocity, respectively).

4) Rotation with constant angular velocity ω (frequency $\nu = \omega/2\pi$) along a circle of radius R:

$$x = R\cos(\theta_0 + \omega\,t), \qquad y = R\sin(\theta_0 + \omega t);$$

$$r = R, \qquad \theta = \theta_0 + \omega\,t$$

[(x, y) and (r, θ) are the Cartesian and polar coordinates of the rotating particle, θ_0 is the initial value of θ].

$$v = R\,\omega, \qquad a = R\,\omega^2$$

(v and a are the linear velocity and acceleration).

5) Rotation with constant angular acceleration α:

$$\omega = \omega_0 + \alpha\, t, \qquad \theta = \theta_0 + \omega_0\, t + \frac{\alpha\, t^2}{2}\,.$$

Dynamics

1) *Newton's Second Law:*

$$m\,\mathbf{a} = \mathbf{F}$$

2) For a sliding body, the friction force is $F_{fr} = \pm k\,N$, where N is the reaction force and the sign is determined by the direction of the axes and geometry of the problem.

3) Conserved quantities:

linear momentum: $\mathbf{P} = m_1\,\mathbf{v}_1 + m_2\,\mathbf{v}_2 + \ldots$

angular momentum
(with respect to the origin): $\mathbf{A}_o = m_1\,\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + \ldots$

angular momentum
(with respect to a point P): $\mathbf{A}_P = m_1\,(\mathbf{r}_1 - \mathbf{r}_P) \times \dot{\mathbf{r}}_1 + \ldots$

energy: $E = \dfrac{m_1\,v_1^2}{2} + \ldots + U(x_1, x_2 \ldots)$,

where U is the *potential* energy.

4) A *conservative* force **F** and the corresponding potential energy U are related by

$$\mathbf{F} = -\nabla U.$$

5) The potential energy and force for a *spring* of modulus k and unperturbed length L are

$$U = \frac{k\,(L' - L)^2}{2}, \qquad F = \pm k\,(L' - L),$$

where L′ is the "current" length of the spring and the sign for F is determined by the direction of the axes and geometry of the problem.

6) The potential energy U and force F for a particle of mass m located at a height H, in the Earth's *gravitational field* are

"locally": $U = -mg\,H,$ $F = -mg,$

"globally": $$U = -\frac{\gamma m_{Earth}\, m}{R_{Earth} + H}, \qquad U = \pm\frac{\gamma m_{Earth}\, m}{(R_{Earth} + H)^2},$$

($\gamma = 6.7 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$, $M_{Earth} = 6.0 \times 10^{24}$ kg, $R_{Earth} = 6.4 \times 10^6$ m).

7) The angular velocity of a body rotating along a circular orbit around a much heavier body of mass m is

$$\omega = \sqrt{\frac{\gamma M}{r^3}},$$

where r is the radius of rotation.

Oscillations

The equation of forced linear pendulum is

$$\ddot{\phi} + 2c\,\dot{\phi} + \omega^2\,\phi = F_0 \cos \Omega t,$$

where c is the friction coefficient, $\omega^2 = L/g$, F_0 and Ω are the amplitude and frequency of the external forcing.

Hamiltonian Mechanics

1) The Hamiltonian equations are

$$\dot{x}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial x_j}, \qquad \text{where } j = 1, 2 \ldots n.$$

2) The Poisson brackets of functions $F(x_1 \ldots x_n, p_1 \ldots p_n)$ and $G(x_1 \ldots x_n, p_1 \ldots p_n)$ are

$$\{F, G\} = \sum_{j=1}^{n} \left(\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial x_j} - \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial p_j} \right).$$

3) A transformation

$$x_i' = x_i'(x_1 \ldots x_n, p_1 \ldots p_n), \qquad p_i' = p_i'(x_1 \ldots x_n, p_1 \ldots p_n),$$

is canonical if and only if

$$\{x_i', p_k'\} = \begin{cases} -1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases} \qquad \{x_i', x_k'\} = \{p_i', p_k'\} = 0.$$

4) The Lagrangian equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j} \right) - \frac{\partial L}{\partial x_j} = 0, \qquad \text{where } j = 1, 2 \ldots n.$$

Stability of Dynamical Systems

Let $\mathbf{x}_F$ be a fixed point of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_k \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} f_1(x_1 \ldots x_k) \\ f_2(x_1 \ldots x_k) \\ \ldots \\ f_k(x_1 \ldots x_k) \end{bmatrix}.$$

Then,

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_k} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_k} \\ \cdots & & & \\ \dfrac{\partial f_k}{\partial x_1} & \dfrac{\partial f_k}{\partial x_2} & \cdots & \dfrac{\partial f_k}{\partial x_k} \end{bmatrix}$$

is the Jacobian matrix of the system at $\mathbf{x}_F$, with $\lambda_1 ... \lambda_k$ being its eigenvalues. Then,

if Re $\lambda_j < 0$ for all j $\Rightarrow$ $\mathbf{x}_F$ is asymptotically stable;

if Re $\lambda_j > 0$ for some j $\Rightarrow$ $\mathbf{x}_F$ is unstable;

if Re $\lambda_j < 0$ for some j, and Re $\lambda_j = 0$ for the remaining j $\Rightarrow$ the test is inconclusive.