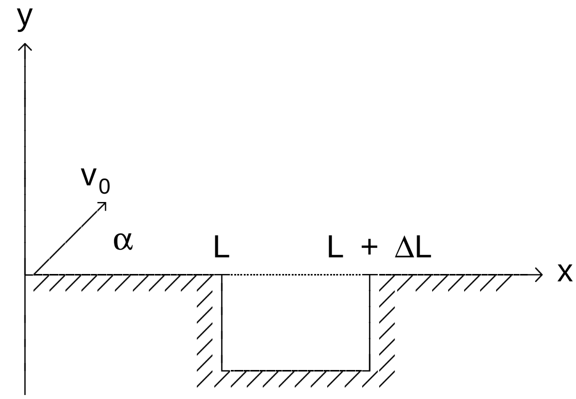
**Question 1** [25 marks]

1) A stone is projected (under gravity, with air friction neglected) with velocity $v_0$, at an angle $\alpha$ towards a "pit" of width $\Delta L$, located at a distance L:




Find for which values of $\alpha$ the stone ends up inside the pit.

2) The trajectory of a particle is given by

$$x = t \cos t, \qquad y = \sin t, \qquad t: 0 \rightarrow 3\pi.$$

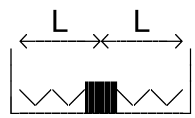
Sketch the trajectory of the particle.

3) The same as in part 2, but for a particle with polar coordinates, given by

$$r = \frac{1}{\cos t}, \quad \theta = t, \quad t: 0 \rightarrow \pi/2.$$

**Question 2** [25 marks]

1) A particle of mass m is attached to two identical springs of module k and unperturbed length L, which are in turn attached to walls:




Initially, the particle's velocity is v. Find how far it will get.

2) A thin stick of mass m and length L is rotating around its end with angular velocity $\omega$. Calculate the stick's kinetic energy?

3) A body of mass m is attached to the ceiling by a spring of unperturbed length L and modulus k.

3.1) Find the distance $H_{eq}$ between the body's equilibrium position and the ceiling.

3.2) Assuming that the drag (friction) force is $F_{dr} = -\mu v$ ($\mu$ is the friction coefficient, v is the velocity of the body), derive an equation for the distance x(t) between the body's current and equilibrium positions.

3.3) Assuming that $m = 2$, $k = 8$, and $\mu = 4$, determine whether the oscillations of the body are underdamped, overdamped, or critically damped. Draw a typical graph of x(t).

4) Consider two particles of masses $m_{1,2}$ separated by a distance L, and an object located on the straight line connecting them. Where should the object be placed so that the gravity forces exerted on it by the particles are in exact balance?

**Question 3** [25 marks]

1) Consider a one-dimensional system which consists of two particles of masses $m_1$ and $m_2$, with coordinates $x_1$ and $x_2$ ($x_1 < x_2$) interacting through gravity. Write down the expression for the Lagrangian of the system, and derive the Lagrangian form of the governing equations.

2) Consider the two-dimensional equivalent of the system described in Question 1 above, i.e. two particles of masses $m_1$ and $m_2$, with coordinates $(x_1,y_1)$ and $(x_2,y_2)$, interacting through gravity.

Write down the expression for the kinetic and potential energies of the system.

3) Find and examine the fixed points of

$$\dot{\phi} = \phi + \psi^2 + \psi, \qquad \dot{\psi} = \phi + \psi^2 + 1.$$

**Question 4** [25 marks]

Consider

$$\ddot{\phi} + 2c\,\dot{\phi} + (1 + \varepsilon \cos 2\Omega t)\,\phi = 0, \qquad (1)$$

where $\varepsilon$, $c \ll 1$ and $\Omega \approx 1$.

1) Seek a solution in the form

$$\phi = B(t) \cos \Omega t + D(t) \sin \Omega t. \qquad (2)$$

2) Upon substitution of (2) into (1), omit small terms involving $\ddot{B}$, $\ddot{D}$, $c\dot{B}$, and $c\dot{D}$.

3) Omit the non-resonant terms, i.e. terms involving $\cos 3\Omega t$ and $\sin 3\Omega t$.

4) Collect like terms and obtain a set of equations for $B(t)$ and $D(t)$. Reduce this set to a single equation for $D(t)$ and find its general solution.

5) On the basis of this solution, determine the range of $\Omega$ for which parametric resonance occurs in the system.

# THEORETICAL MECHANICS (SUMMARY)

**Kinematics**

1) Position vector **r**, velocity **v**, and acceleration **a** of a particle:

$$\mathbf{v} = \dot{\mathbf{r}}, \qquad \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}.$$

2) 1D motion with constant velocity v:

$$x = x_0 + v\,t.$$

3) 1D motion with constant acceleration a:

$$v = v_0 + a\,t, \qquad x = x_0 + v_0\,t + \frac{a\,t^2}{2}$$

($x_0$ and $v_0$ are the initial coordinate and velocity, respectively).

4) Rotation with constant angular velocity $\omega$ (frequency $\nu = {}^{\omega}/2\pi$) along a circle of radius R:

$$x = R\cos(\theta_0 + \omega\,t), \qquad y = R\sin(\theta_0 + \omega t);$$

$$r = R, \qquad \theta = \theta_0 + \omega\,t$$

[(x, y) and (r, $\theta$) are the Cartesian and polar coordinates of the rotating particle, $\theta_0$ is the initial value of $\theta$].

$$v = R\,\omega, \qquad a = R\,\omega^2$$

(v and a are the linear velocity and acceleration).

5) Rotation with constant angular acceleration $\alpha$:

$$\omega = \omega_0 + \alpha\, t, \qquad \theta = \theta_0 + \omega_0\, t + \frac{\alpha\, t^2}{2}\,.$$

**Dynamics**

1) *Newton's Second Law:*

$$m\,\mathbf{a} = \mathbf{F}$$

2) For a sliding body, the friction force is $F_{fr} = \pm k\, N$, where N is the reaction force and the sign is determined by the direction of the axes and geometry of the problem.

3) Conserved quantities:

*linear momentum*: $\mathbf{P} = m_1\,\mathbf{v}_1 + m_2\,\mathbf{v}_2 + \dots$

*angular momentum*
(with respect to the origin): $\mathbf{A}_o = m_1\,\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + \dots$

*angular momentum*
(with respect to a point P): $\mathbf{A}_P = m_1\,(\mathbf{r}_1 - \mathbf{r}_P) \times \dot{\mathbf{r}}_1 + \dots$

*energy*: $E = \dfrac{m_1\, v_1^2}{2} + \dots + U(x_1, x_2 \dots),$

where U is the *potential* energy.

4) A *conservative* force **F** and the corresponding potential energy U are related by

$$\mathbf{F} = -\nabla U.$$

5) The potential energy and force for a *spring* of modulus k and unperturbed length L are

$$U = \frac{k\,(L' - L)^2}{2}\,, \qquad F = \pm k\,(L' - L),$$

where $L'$ is the "current" length of the spring and the sign for F is determined by the direction of the axes and geometry of the problem.

6) The potential energy U and force F for a particle of mass m located at a height H, in the Earth's *gravitational field* are

"locally": $U = -mg\,H,$ $F = -mg,$

"globally": $U = -\dfrac{\gamma m_{Earth}\, m}{R_{Earth} + H}\,,$ $U = \pm\dfrac{\gamma m_{Earth}\, m}{(R_{Earth} + H)^2}\,,$

($\gamma = 6.7 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$, $M_{Earth} = 6.0 \times 10^{24}$ kg, $R_{Earth} = 6.4 \times 10^6$ m).

7) The angular velocity of a body rotating along a circular orbit around a much heavier body of mass m is

$$\omega = \sqrt{\frac{\gamma M}{r^3}}\,,$$

where r is the radius of rotation.

**Oscillations**

The equation of forced linear pendulum is

$$\ddot{\phi} + 2c\,\dot{\phi} + \omega^2\,\phi = F_0 \cos \Omega t,$$

where c is the friction coefficient, $\omega^2 = L/g$, $F_0$ and $\Omega$ are the amplitude and frequency of the external forcing.

**Hamiltonian Mechanics**

1) The Hamiltonian equations are

$$\dot{x}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial x_j}, \qquad \text{where} \quad j = 1, 2 \dots n.$$

2) The Poisson brackets of functions $F(x_1 \dots x_n, p_1 \dots p_n)$ and $G(x_1 \dots x_n, p_1 \dots p_n)$ are

$$\{F, G\} = \sum_{j=1}^{n} \left( \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial x_j} - \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial p_j} \right).$$

3) A transformation

$$x_i' = x_i'(x_1 \dots x_n, p_1 \dots p_n), \qquad p_i' = p_i'(x_1 \dots x_n, p_1 \dots p_n),$$

is canonical if and only if

$$\{x_i', p_k'\} = \begin{cases} -1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases} \qquad \{x_i', x_k'\} = \{p_i', p_k'\} = 0.$$

4) The Lagrangian equations are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_j} \right) - \frac{\partial L}{\partial x_j} = 0, \qquad \text{where } j = 1, 2 \dots n.$$

**Stability of Dynamical Systems**

Let $\mathbf{x}_F$ be a fixed point of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_k \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} f_1(x_1 \dots x_k) \\ f_2(x_1 \dots x_k) \\ \dots \\ f_k(x_1 \dots x_k) \end{bmatrix}.$$

Then,

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_k} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_k} \\ \cdots & & & \\ \dfrac{\partial f_k}{\partial x_1} & \dfrac{\partial f_k}{\partial x_2} & \cdots & \dfrac{\partial f_k}{\partial x_k} \end{bmatrix}$$

is the Jacobian matrix of the system at $\mathbf{x}_F$, with $\lambda_1 ... \lambda_k$ being its eigenvalues. Then,

if Re $\lambda_j < 0$ for all j $\Rightarrow$ $\mathbf{x}_F$ is asymptotically stable;

if Re $\lambda_j > 0$ for some j $\Rightarrow$ $\mathbf{x}_F$ is unstable;

if Re $\lambda_j < 0$ for some j, and Re $\lambda_j = 0$ for the remaining j $\Rightarrow$ the test is inconclusive.