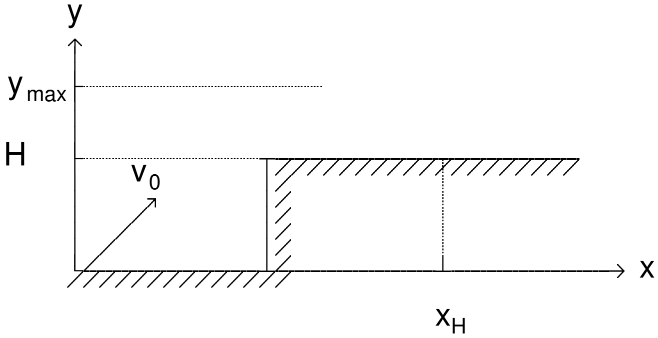
Question 1 [25 marks]

1) A stone is projected (under gravity, with air friction neglected) with velocity v_0, at an angle α to the horizon, towards a "step" of height H:



1.1) Assuming that the stone goes over the step, calculate y_{max} (the maximum height of the stone's trajectory) and x_H (the x-coordinate of the point where it hits the ground).

1.2) Determine for which v_0 the stone would go over the step.

2) The trajectory of a particle is given by

$$x = 3 \cos t, \qquad y = \sin t, \qquad t\colon 0 \rightarrow \tfrac{3}{4}\pi.$$

Sketch the trajectory of the particle.

3) The same as in part 2, but for a particle with polar coordinates, given by

$$r = 2\pi - t, \qquad \theta = t, \qquad t\colon 0 \rightarrow 2\pi.$$

Question 2 [25 marks]

1) Three particles, of masses 1 kg, 3 kg, and 2 kg, simultaneously collide:

Before the collision, the middle particle was motionless, whereas the velocities of the other two were 1 m/s and -3 m/s (see the diagram). Assuming that the particles collide non-elastically and coalesce, find their velocity after the collision. Which way will they be moving?

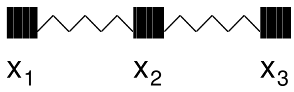
2) Consider a system of 3 particles of masses m_1, m_2, and m_3, with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, interacting with forces $\mathbf{F}_{1,2}$, $\mathbf{F}_{2,1}$ ... $\mathbf{F}_{3,2}$. Prove that the angular momentum of the system with respect to the origin is conserved.

3) The velocity of a particle, which has slid down a plane tilted at an angle α, is v. Assuming that the friction coefficient is k, find the height from which the particle started its motion.

4) Two spherical objects, of radii $R_{1,2}$ and masses $m_{1,2}$, are attracted to each other through gravity. The initial velocities of the objects are zero, the initial distance separating them is infinitely large. Find their velocities when they collide.

Question 3 [20 marks]

1) Consider a one-dimensional system which consists of three particles of masses m_1, m_2, and m_3, with coordinates x_1, x_2, and x_3 ($x_1 \le x_2 \le x_3$) connected by two identical springs of modulus μ and free length L:



1.1) Write down the expression for the Hamiltonian H of this system.

1.2) Write down the Hamiltonian equations for this system.

1.3) Write down the expression for the momentum P of this system.

1.4) Prove that P is conserved.

1.5) Write down the expression for the Lagrangian L of the system, and derive the Lagrangian form of the governing equations.

2) Find and examine the fixed points of

$$\dot{\phi} = -\psi, \qquad \dot{\psi} = \phi^2 - \phi\,\psi - 1.$$

Question 4 [30 marks]

Consider

$$\ddot{\phi} + 2c\,\dot{\phi} + (1 + \varepsilon \cos 2\Omega t)\,\phi = 0, \tag{1}$$

where ε, $c \ll 1$ and $\Omega \approx 1$.

1) Seek a solution in the form

$$\phi = B(t) \cos \Omega t + D(t) \sin \Omega t. \tag{2}$$

2) Upon substitution of (2) into (1), omit small terms involving $\ddot{B}$, $\ddot{D}$, $c\dot{B}$, and $c\dot{D}$.

3) Omit the non-resonant terms, i.e. terms involving $\cos 3\Omega t$ and $\sin 3\Omega t$.

4) Collect like terms and solve the resulting set of equations for $B(t)$ and $D(t)$.

5) Using these equations, determine the range of Ω for which parametric resonance occurs in the system.

THEORETICAL MECHANICS (SUMMARY)

Kinematics

1) Position vector **r**, velocity **v**, and acceleration **a** of a particle:

$$\mathbf{v} = \dot{\mathbf{r}}, \qquad \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}.$$

2) 1D motion with constant velocity v:

$$x = x_0 + v\,t.$$

3) 1D motion with constant acceleration a:

$$v = v_0 + a\,t, \qquad x = x_0 + v_0\,t + \frac{a\,t^2}{2}$$

(x_0 and v_0 are the initial coordinate and velocity, respectively).

4) Rotation with constant angular velocity ω (frequency $\nu = \omega/2\pi$) along a circle of radius R:

$$x = R\cos(\theta_0 + \omega\,t), \qquad y = R\sin(\theta_0 + \omega t);$$

$$r = R, \qquad \theta = \theta_0 + \omega\,t$$

[(x, y) and (r, θ) are the Cartesian and polar coordinates of the rotating particle, θ_0 is the initial value of θ].

$$v = R\,\omega, \qquad a = R\,\omega^2$$

(v and a are the linear velocity and acceleration).

5) Rotation with constant angular acceleration α:

$$\omega = \omega_0 + \alpha\, t, \qquad \theta = \theta_0 + \omega_0\, t + \frac{\alpha\, t^2}{2}.$$

Dynamics

1) *Newton's Second Law:*

$$m\,\mathbf{a} = \mathbf{F}$$

2) For a sliding body, the friction force is $F_{fr} = \pm k\, N$, where N is the reaction force and the sign is determined by the direction of the axes and geometry of the problem.

3) Conserved quantities:

linear momentum: $\mathbf{P} = m_1\,\mathbf{v}_1 + m_2\,\mathbf{v}_2 + \ldots$

angular momentum
(with respect to the origin): $\mathbf{A}_o = m_1\,\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + \ldots$

angular momentum
(with respect to a point P): $\mathbf{A}_P = m_1\,(\mathbf{r}_1 - \mathbf{r}_P) \times \dot{\mathbf{r}}_1 + \ldots$

energy: $E = \dfrac{m_1\, v_1^2}{2} + \ldots + U(x_1, x_2 \ldots),$

where U is the *potential* energy.

4) A *conservative* force **F** and the corresponding potential energy U are related by

$$\mathbf{F} = -\nabla U.$$

5) The potential energy and force for a *spring* of modulus k and unperturbed length L are

$$U = \frac{k\,(L' - L)^2}{2}, \qquad F = \pm k\,(L' - L),$$

where L' is the "current" length of the spring and the sign for F is determined by the direction of the axes and geometry of the problem.

6) The potential energy U and force F for a particle of mass m located at a height H, in the Earth's *gravitational field* are

"locally": $U = -mg\,H,\qquad F = -mg,$

"globally": $U = -\dfrac{\gamma m_{Earth}\, m}{R_{Earth} + H},\qquad U = \pm\dfrac{\gamma m_{Earth}\, m}{(R_{Earth} + H)^2},$

($\gamma = 6.7 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$, $M_{Earth} = 6.0 \times 10^{24}$ kg, $R_{Earth} = 6.4 \times 10^6$ m).

7) The angular velocity of a body rotating along a circular orbit around a much heavier body of mass m is

$$\omega = \sqrt{\frac{\gamma M}{r^3}},$$

where r is the radius of rotation.

Oscillations

The equation of forced linear pendulum is

$$\ddot{\phi} + 2c\,\dot{\phi} + \omega^2\,\phi = F_0 \cos \Omega t,$$

where c is the friction coefficient, $\omega^2 = L/g$, F_0 and Ω are the amplitude and frequency of the external forcing.

Hamiltonian Mechanics

1) The Hamiltonian equations are

$$\dot{x}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial x_j}, \qquad \text{where } j = 1, 2 \ldots n.$$

2) The Poisson brackets of functions $F(x_1 \ldots x_n, p_1 \ldots p_n)$ and $G(x_1 \ldots x_n, p_1 \ldots p_n)$ are

$$\{F, G\} = \sum_{j=1}^{n} \left(\frac{\partial F}{\partial p_j} \frac{\partial G}{\partial x_j} - \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial p_j} \right).$$

3) A transformation

$$x_i' = x_i'(x_1 \ldots x_n, p_1 \ldots p_n), \qquad p_i' = p_i'(x_1 \ldots x_n, p_1 \ldots p_n),$$

is canonical if and only if

$$\{x_i', p_k'\} = \begin{cases} -1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases} \qquad \{x_i', x_k'\} = \{p_i', p_k'\} = 0.$$

4) The Lagrangian equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j} \right) - \frac{\partial L}{\partial x_j} = 0, \qquad \text{where } j = 1, 2 \ldots n.$$

Stability of Dynamical Systems

Let $\mathbf{x}_F$ be a fixed point of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_k \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} f_1(x_1 \ldots x_k) \\ f_2(x_1 \ldots x_k) \\ \ldots \\ f_k(x_1 \ldots x_k) \end{bmatrix}.$$

Then,

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_k} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_k} \\ \cdots & & & \\ \dfrac{\partial f_k}{\partial x_1} & \dfrac{\partial f_k}{\partial x_2} & \cdots & \dfrac{\partial f_k}{\partial x_k} \end{bmatrix}$$

is the Jacobian matrix of the system at $\mathbf{x}_F$, with $\lambda_1 \ldots \lambda_k$ being its eigenvalues. Then,

if Re $\lambda_j < 0$ for all j $\Rightarrow$ $\mathbf{x}_F$ is asymptotically stable;

if Re $\lambda_j > 0$ for some j $\Rightarrow$ $\mathbf{x}_F$ is unstable;

if Re $\lambda_j < 0$ for some j, and Re $\lambda_j = 0$ for the remaining j $\Rightarrow$ the test is inconclusive.