Question 1 [30 marks]

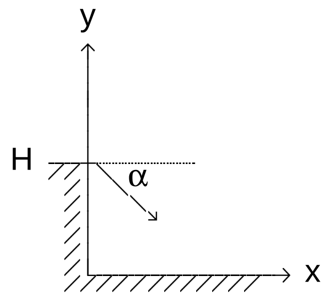
1) A cyclist sets out with velocity v_0 = 30 km/h, and decelerates with a = -10 km/h^2 until he stops. He rests for 1 hour, then cycles in the opposite direction for 4 hours with constant velocity v_1 = 20 km/h.

a) Draw the graphs of the velocity and displacement of the cyclist vs. time.

b) Find how far the cyclist ends up from his starting point.

c) Find the total distance covered by the cyclist and his mean velocity.

2) A stone is thrown with velocity v_0, at an angle α to the horizontal, from a "step" of height H:



Calculate the x-coordinate of the point where the stone hits the ground.

3) The Cartesian coordinates of a particle on the plane are given by

$$x = t \cos t, \qquad y = -t \sin t, \qquad t: 0 \rightarrow \pi.$$

Find the velocity and acceleration of the particle, and sketch its trajectory (show as much detail as possible).

4) Sketch the trajectory of a particle with polar coordinates, given by

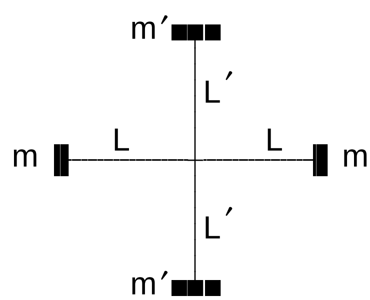
$$r = 1 + t, \qquad \theta = -2t, \qquad t: 0 \rightarrow \pi.$$

Question 2 [25 marks]

1) A particle is pushed, with initial velocity v_0, up a slope of angle α. The friction coefficient is k. Find how far the particle will get before stopping.

2) A sling L metres long, with a stone of mass m kilograms, is being rotated with frequency ν revolutions per second. Find the linear velocity of the stone and the strain (force) in the sling's handles.

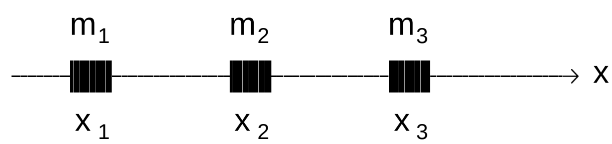
3) The following system of four interconnected particles, of masses m and m′ as shown on the diagram,



is rotating around its centre. Then, the distance L is reduced by a factor of 2. Using conservation of angular momentum, determine how L′ needs to be changed to ensure that the angular velocity of the system remains the same as before.

Question 3 [25 marks]

Consider a one-dimensional system which consists of three particles of masses m_1, m_2, and m_3, with coordinates x_1, x_2, and x_3 ($x_1 \le x_2 \le x_3$), interacting with one another through gravity.



1) Using the one-dimensional version of Newton's Law of Gravity, determine the forces F_{12}, F_{21}, F_{13}, F_{31}, F_{23}, F_{32}, where F_{ij} is the force exerted by the j-th particle on the i-th particle.

2) Using the relationship between the potential energy U of the system and the corresponding forces, show that the above expressions correspond to

$$U(x_1, x_2, x_3) = -\frac{\gamma m_1 m_2}{x_2 - x_1} - \frac{\gamma m_2 m_3}{x_3 - x_2} - \frac{\gamma m_3 m_1}{x_3 - x_1} .$$

3) Write down the expression for the Hamiltonian H of the system.

4) Write down the Hamiltonian equations for this system.

5) Write down the expression for the momentum P of this system.

6) Show that P is an integral of motion (conserved quantity).

7) Write down the expression for the Lagrangian L of the system, and derive the Lagrangian form of the governing equations.

Question 4 [20 marks]

Find and examine the fixed points of

$$\dot{\phi} = \psi, \qquad \dot{\psi} = \phi^2 + \phi\,\psi - 1.$$

THEORETICAL MECHANICS (SUMMARY)

Kinematics

1) Position vector **r**, velocity **v**, and acceleration **a** of a particle:

$$\mathbf{v} = \dot{\mathbf{r}}, \qquad \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}.$$

2) 1D motion with constant velocity v:

$$x = x_0 + v\,t.$$

3) 1D motion with constant acceleration a:

$$v = v_0 + a\,t, \qquad x = x_0 + v_0\,t + \frac{a\,t^2}{2}$$

(x_0 and v_0 are the initial coordinate and velocity, respectively).

4) Rotation with constant angular velocity ω (frequency $\nu = \omega/2\pi$) along a circle of radius R:

$$x = R\cos(\theta_0 + \omega\,t), \qquad y = R\sin(\theta_0 + \omega t);$$

$$r = R, \qquad \theta = \theta_0 + \omega\,t$$

[(x, y) and (r, θ) are the Cartesian and polar coordinates of the rotating particle, θ_0 is the initial value of θ].

$$v = R\,\omega, \qquad a = R\,\omega^2$$

(v and a are the linear velocity and acceleration).

5) Rotation with constant angular acceleration α:

$$\omega = \omega_0 + \alpha\, t, \qquad \theta = \theta_0 + \omega_0\, t + \frac{\alpha\, t^2}{2}\,.$$

Dynamics

1) *Newton's Second Law:*

$$m\,\mathbf{a} = \mathbf{F}$$

2) For a sliding body, the friction force is $F_{fr} = \pm k\, N$, where N is the reaction force and the sign is determined by the direction of the axes and geometry of the problem.

3) Conserved quantities:

linear momentum: $\mathbf{P} = m_1\,\mathbf{v}_1 + m_2\,\mathbf{v}_2 + \ldots$

angular momentum (with respect to the origin): $\mathbf{A}_o = m_1\,\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + \ldots$

angular momentum (with respect to a point P): $\mathbf{A}_P = m_1\,(\mathbf{r}_1 - \mathbf{r}_P) \times \dot{\mathbf{r}}_1 + \ldots$

energy: $E = \dfrac{m_1\, v_1^2}{2} + \ldots + U(x_1,x_2\ldots),$

where U is the *potential* energy.

4) A *conservative* force **F** and the corresponding potential energy U are related by

$$\mathbf{F} = -\nabla U.$$

5) The potential energy and force for a *spring* of modulus k and unperturbed length L are

$$U = \frac{k\,(L' - L)^2}{2}\,, \qquad F = \pm k\,(L' - L),$$

where L' is the "current" length of the spring and the sign for F is determined by the direction of the axes and geometry of the problem.

6) The potential energy U and force F for a particle of mass m located at a height H, in the Earth's *gravitational field* are

"locally": $U = -mg\, H, \qquad F = -mg,$

"globally": $U = -\dfrac{\gamma m_{Earth}\, m}{R_{Earth} + H}, \qquad U = \pm \dfrac{\gamma m_{Earth}\, m}{(R_{Earth} + H)^2},$

($\gamma = 6.7 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$, $M_{Earth} = 6.0 \times 10^{24}$ kg, $R_{Earth} = 6.4 \times 10^6$ m).

7) The angular velocity of a body rotating along a circular orbit around a much heavier body of mass m is

$$\omega = \sqrt{\frac{\gamma M}{r^3}},$$

where r is the radius of rotation.

Oscillations

The equation of forced linear pendulum is

$$\ddot{\phi} + 2c\,\dot{\phi} + \omega^2\,\phi = F_0 \cos \Omega t,$$

where c is the friction coefficient, $\omega^2 = L/g$, F_0 and Ω are the amplitude and frequency of the external forcing.

Hamiltonian Mechanics

1) The Hamiltonian equations are

$$\dot{x}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial x_j}, \qquad \text{where } j = 1, 2... n.$$

2) The Poisson brackets of functions $F(x_1...x_n,p_1...p_n)$ and $G(x_1...x_n,p_1...p_n)$ are

$$\{F, G\} = \sum_{j=1}^{n} \left(\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial x_j} - \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial p_j} \right).$$

3) A transformation

$$x'_i = x'_i(x_1...x_n,p_1...p_n), \qquad p'_i = p'_i(x_1...x_n,p_1...p_n),$$

is canonical if and only if

$$\{x'_i, p'_k\} = \begin{cases} -1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases} \qquad \{x'_i, x'_k\} = \{p'_i, p'_k\} = 0.$$

4) The Lagrangian equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j} = 0, \qquad \text{where } j = 1, 2... n.$$

Stability of Dynamical Systems

Let $\mathbf{x}_F$ be a fixed point of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_k \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} f_1(x_1...x_k) \\ f_2(x_1...x_k) \\ ... \\ f_k(x_1...x_k) \end{bmatrix}.$$

Then,

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_k} \\ \cdots & & & \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_k} \end{bmatrix}$$

is the Jacobian matrix of the system at $\mathbf{x}_F$, with $\lambda_1 ... \lambda_k$ being its eigenvalues. Then,

if $\mathrm{Re}\,\lambda_j < 0$ for all j $\Rightarrow$ $\mathbf{x}_F$ is asymptotically stable;

if $\mathrm{Re}\,\lambda_j > 0$ for some j $\Rightarrow$ $\mathbf{x}_F$ is unstable;

if $\mathrm{Re}\,\lambda_j < 0$ for some j, and $\mathrm{Re}\,\lambda_j = 0$ for the remaining j $\Rightarrow$ the test is inconclusive.