MS4414 Theoretical Mechanics
Tutorial week10: Angular Momentum

Thursday 31 March 2011

Angular momentum

- Definition for a single particle

$$\mathbf{L} = m\mathbf{r} \times \mathbf{v} = m\mathbf{r} \times \dot{\mathbf{r}}$$

- Definition for a group of particles

$$\mathbf{L} = \sum_i m_i \mathbf{r_i} \times \mathbf{v_i} = \sum_i m_i \mathbf{r_i} \times \dot{\mathbf{r}}_\mathbf{i}$$

- Prove the conservation: calculate

$$\frac{d\mathbf{L}}{dt}$$

- Prove this value is $\mathbf{0}$ using
  - The system is closed $(\mathbf{F_{12}} = -\mathbf{F_{21}})$
  - Differentiation

$$\frac{d(\mathbf{a} \times \mathbf{b})}{dt} = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}$$

Question 1

- Original momentum

$$\mathbf{M} = 2mL^2\omega + 2m'L'^2\omega$$

- New momentum

$$\mathbf{M_1} = 2mL_1^2\omega + 2m'L_1'^2\omega$$

- Relationship $L_1 = L/2$

$$\mathbf{M} = \mathbf{M_1} \Longrightarrow \frac{L_1'}{L'} = \sqrt{1 + \frac{3}{4}\frac{m}{m'}\frac{L^2}{L'^2}}$$

Question 2

- Calculate the angular momentum

$$\mathbf{L} = m_1\mathbf{r_1} \times \mathbf{v_1} + m_2\mathbf{r_2} \times \mathbf{v_2} + m_3\mathbf{r_3} \times \mathbf{v_3}$$

- Calculate the derivatives

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= m_1\mathbf{r_1} \times \mathbf{a_1} + m_2\mathbf{r_2} \times \mathbf{a_2} + m_3\mathbf{r_3} \times \mathbf{a_3} \\ &= \mathbf{r_1} \times (\mathbf{F_{12}} + \mathbf{F_{13}}) + \mathbf{r_2} \times (\mathbf{F_{21}} + \mathbf{F_{23}}) \\ &\quad + \mathbf{r_3} \times (\mathbf{F_{31}} + \mathbf{F_{32}}) \\ &= (\mathbf{r_1} - \mathbf{r_2}) \times \mathbf{F_{12}} + (\mathbf{r_2} - \mathbf{r_3}) \times \mathbf{F_{23}} + (\mathbf{r_3} - \mathbf{r_1}) \times \mathbf{F_{31}} \\ &= \mathbf{0} \end{aligned}$$

Question 3

- Momentum

$$\mathbf{P} = m_1 \dot{\mathbf{r_1}} + m_2 \dot{\mathbf{r_2}}$$

- Derivative

$$\frac{d\mathbf{P}}{dt} = m_1 \mathbf{a_1} + m_2 \mathbf{a_2} = \mathbf{F_{12}} + \mathbf{F_{21}} = \mathbf{0}$$

- Angular momentum

$$\mathbf{A_M} = m_1 \left(\mathbf{r_1} - \mathbf{r_M}\right) \times \mathbf{v_1} + m_2 \left(\mathbf{r_2} - \mathbf{r_M}\right) \times \mathbf{v_2}$$

- Derivative of angular momentum

$$\frac{d\mathbf{A_M}}{dt} = \left(\mathbf{r_1} - \mathbf{r_2}\right) \times \mathbf{F_{12}} = \mathbf{0}$$