Theoretical Mechanics: Summary

Kinematics

1. Position vector $\mathbf{r}$, velocity $\mathbf{v}$, and acceleration $\mathbf{a}$ of a particle are related by:

$$\mathbf{v} = \dot{\mathbf{r}}, \qquad \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}.$$

2. 1D motion with constant velocity v:

$$x = vt + x_0.$$

3. 1D motion with constant acceleration a:

$$v = at + v_0, \qquad x = \frac{1}{2}at^2 + v_0 t + x_0,$$

   where x_0 and v_0 are the position and velocity at $t = 0$, respectively.

4. Rotation with constant angular velocity ω (frequency $\nu = \frac{\omega}{2\pi}$) along a circle of radius R:

   - polar coordinates

$$r = R, \qquad \theta = \omega t + \theta_0,$$

   - Cartesian coordinates

$$x = R\cos(\omega t + \theta_0)\,, \qquad y = R\sin(\omega t + \theta_0)\,,$$

   - linear velocity

$$v = R\omega,$$

   - acceleration

$$a = R\omega^2,$$

   where θ_0 is the value of θ at $t = 0$.

5. Rotation with constant angular acceleration α:

$$\omega = \alpha t + \omega_0, \qquad \theta = \tfrac{1}{2}\alpha t^2 + \omega_0 t + \theta_0,$$

where ω is the angular velocity, θ is the angular coordinate, ω_0 is the angular velocity at $t = 0$, and θ_0 is the angular coordinate at $t = 0$.

Dynamics

1. Newton's second law:

$$m\mathbf{a} = \mathbf{F}.$$

2. For a sliding body, the friction force is $F_{\text{rf}} = kN$, when N is the normal reaction force. It is oriented in the opposite sense of the motion.

3. Conserved quantities:

   - linear momentum

$$\mathbf{P} = \sum_i m_i \mathbf{v}_i,$$

   - angular momentum with respect to the origin

$$\mathbf{A}_O = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i,$$

   - angular momentum with respect to an arbitrary point P

$$\mathbf{A}_P = \sum_i m_i \left(\mathbf{r}_i - \mathbf{r}_P\right) \times \dot{\mathbf{r}}_i,$$

   - total energy

$$E = U(x_1, x_2, \ldots) + \sum_i \tfrac{1}{2} m_i v_i^2,$$

   where U is the potential energy.

4. A conservative force $\mathbf{F}$ and the corresponding potential energy U are related by

$$\mathbf{F} = -\boldsymbol{\nabla} U.$$

5. The potential energy and force for a spring of modulus k, and unperturbed length L_0 are

$$U = \tfrac{1}{2}k\left(L - L_0\right)^2, \qquad F = -k\left(L - L_0\right),$$

where L is the current length of the spring. The direction of $\mathbf{F}$ is such that it tries to bring the spring back to its unperturbed configuration.

6. The potential energy U and force F for a particle of mass m located at a height H, in the Earth's gravitational field are

   - locally:

$$U = -mgH, \qquad \mathbf{F} = m\mathbf{g},$$

   - globally:

$$U = -\frac{GM_{\mathrm{E}}m}{R_{\mathrm{E}} + H}, \qquad \mathbf{F} = -\frac{GM_{\mathrm{E}}m}{\left(R_{\mathrm{E}} + H\right)^2}\mathbf{e}_r,$$

   where G is the gravitational constant, M_{E} is the Earth's mass, R_{E} is the Earth's radius, $\mathbf{e}_r$ is a radial unit vector. $G = 6.7 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$, $M_{\mathrm{E}} = 6.0 \times 10^{24}\,\mathrm{kg}$, and $R_{\mathrm{E}} = 6.4 \times 10^6\,\mathrm{m}$.

   - The angular velocity of a body rotating along a circular orbit around a much heavier body of mass M is

$$\omega = \sqrt{\frac{GM}{R^3}}$$

   where R is the orbit radius.

Oscillations

The equation of a forced linear pendulum with small amplitude is

$$\ddot{\phi} + 2c\dot{\phi} + \omega^2\phi = F_0\cos(\Omega_0 t),$$

where c is the friction coefficient, $\omega^2 = \frac{L}{g}$ is the natural frequency of the pendulum, L is the length of the pendulum, F_0 and Ω_0 are the amplitude and frequency of the external forcing.

Hamiltonian mechanics

1. The Hamiltonian equations are

$$\dot{x}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial x_j}, \qquad 1 \le j \le n.$$

2. The Poisson brackets of functions

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x_i} - \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial p_i} \right).$$

3. A transformation

$$x_i' = x_i'(x_1, \dots, x_n, p_1, \dots, p_n), \qquad p_i' = p_i'(x_1, \dots, x_n, p_1, \dots, p_n),$$

   is canonical if and only if

$$\{x_i', p_k'\} = -\delta_{ik}, \qquad \{x_i', x_k'\} = 0, \qquad \{p_i', p_k'\} = 0.$$

4. The Lagrangian equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_j} \right) - \frac{\partial L}{\partial x_j} = 0, \qquad 1 \le j \le n.$$

Stability of dynamical systems

Let $\mathbf{x}_\mathrm{F}$ be a fixed point of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} f_1(x_1, \dots, x_k) \\ f_2(x_1, \dots, x_k) \\ \vdots \\ f_k(x_1, \dots, x_k) \end{bmatrix}.$$

Then,

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_k} \end{bmatrix},$$

is the Jacobian matrix of the system at $\mathbf{x}_\mathrm{F}$, with eigenvalues λ_1, λ_2, …, λ_k.

- If $\Re(\lambda_j) < 0$ for all j then $\mathbf{x}_\mathrm{F}$ is asymptotically stable.
- If $\Re(\lambda_j) > 0$ for some j then $\mathbf{x}_\mathrm{F}$ is unstable.
- If $\Re(\lambda_j) < 0$ for some j, and $\Re(\lambda_j) = 0$ for the remaining j, then the test is inconclusive.