MS4414 Theoretical Mechanics
Tutorial week 11: Lagrangian and Hamiltonian Mechanics

Thursday 7 April 2011

Kinetic and potential energy

- Kinetic energy for a single particle

$$KE = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

- Kinetic energy for two particles

$$KE = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}$$

- Potential energy gravity

$$PE = -\frac{Gm_1m_2}{r_2 - r_1}$$

- Potential energy spring

$$PE = \frac{1}{2}k(x - L)^2$$

- Force

$$F = -\frac{\partial PE}{\partial x}$$

Lagrangian and Hamiltonian

- Lagrangian

$$L = KE(x, \dot{x}) - PE(x, \dot{x})$$

- Lagrangian equations for each x_i

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$

- Hamiltonian

$$L = KE(x, p) + PE(x, p)$$

- Hamiltonian equations

$$\dot{x} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial x}$$

Question 1 (1)

- Forces

$$F_{12} = -F_{21} = \frac{\gamma m_1 m_2}{(x_2 - x_1)^2}\mathbf{u_{12}}$$
$$F_{13} = -F_{31} = \frac{\gamma m_1 m_3}{(x_3 - x_1)^2}\mathbf{u_{13}}$$
$$F_{23} = -F_{32} = \frac{\gamma m_2 m_3}{(x_2 - x_3)^2}\mathbf{u_{23}}$$

- Forces

$$F_1 = -\frac{\partial U}{\partial x_1} \qquad F_2 = -\frac{\partial U}{\partial x_2} \qquad F_3 = -\frac{\partial U}{\partial x_3}$$

- Hamiltonian

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} - \frac{\gamma m_1 m_2}{x_2 - x_1} - \frac{\gamma m_1 m_3}{x_3 - x_1} - \frac{\gamma m_2 m_3}{x_3 - x_2}$$

Question 1 (2)

- Hamiltonian equations

$$
\begin{aligned}
\dot{x}_1 &= \frac{\partial H}{\partial p_1} &\implies& \quad \dot{x}_1 = \frac{p_1}{m_1}\\
\dot{x}_2 &= \frac{\partial H}{\partial p_2} &\implies& \quad \dot{x}_2 = \frac{p_2}{m_2}\\
\dot{x}_3 &= \frac{\partial H}{\partial p_3} &\implies& \quad \dot{x}_3 = \frac{p_3}{m_3}\\
\dot{p}_1 &= -\frac{\partial H}{\partial x_1} &\implies& \quad \dot{p}_1 = -\frac{\gamma m_1 m_2}{(x_2-x_1)^2} - \frac{\gamma m_1 m_3}{(x_3-x_1)^2}\\
\dot{p}_2 &= -\frac{\partial H}{\partial x_2} &\implies& \quad \dot{p}_2 = \frac{\gamma m_1 m_2}{(x_2-x_1)^2} + \frac{\gamma m_2 m_3}{(x_2-x_3)^2}\\
\dot{p}_3 &= -\frac{\partial H}{\partial x_3} &\implies& \quad \dot{p}_3 = \frac{\gamma m_1 m_3}{(x_3-x_1)^2} - \frac{\gamma m_2 m_3}{(x_2-x_3)^2}
\end{aligned}
$$

Question 1 (3)

- Momentum

$$p = p_1 + p_2 + p_3 = m_1\dot{x}_1 + m_2\dot{x}_2 + m_3\dot{x}_3$$

- Closed system $\dot{p} = 0$
- Lagrangian

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{\gamma m_1 m_2}{x_2 - x_1} + \frac{\gamma m_1 m_3}{x_3 - x_1} + \frac{\gamma m_2 m_3}{x_3 - x_2}$$

- Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1} \implies m_1\ddot{x}_1 = -\frac{\gamma m_1 m_2}{(x_2-x_1)^2} - \frac{\gamma m_1 m_3}{(x_3-x_1)^2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = \frac{\partial L}{\partial x_2} \implies m_2\ddot{x}_2 = \frac{\gamma m_1 m_2}{(x_2-x_1)^2} + \frac{\gamma m_2 m_3}{(x_2-x_3)^2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) = \frac{\partial L}{\partial x_3} \implies m_2\ddot{x}_2 = \frac{\gamma m_1 m_3}{(x_3-x_1)^2} - \frac{\gamma m_2 m_3}{(x_2-x_3)^2}$$

Question 2 (1)

- Energies

$$
\begin{aligned}
KE &= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} \\
PE &= \frac{1}{2}k(x_2 - x_1 - L)^2 + \frac{1}{2}k(x_3 - x_2 - L)^2
\end{aligned}
$$

- Hamiltonian

$$
\begin{aligned}
H &= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} \\
&\quad + \frac{1}{2}k(x_2 - x_1 - L)^2 + \frac{1}{2}k(x_3 - x_2 - L)^2
\end{aligned}
$$

Question 2 (2)

- Hamiltonian equations

$$\begin{aligned}
\dot{x}_1 &= \frac{\partial H}{\partial p_1} &\Longrightarrow\quad \dot{x}_1 &= \frac{p_1}{m_1}\\
\dot{x}_2 &= \frac{\partial H}{\partial p_2} &\Longrightarrow\quad \dot{x}_2 &= \frac{p_2}{m_2}\\
\dot{x}_3 &= \frac{\partial H}{\partial p_3} &\Longrightarrow\quad \dot{x}_3 &= \frac{p_3}{m_3}\\
\dot{p}_1 &= -\frac{\partial H}{\partial x_1} &\Longrightarrow\quad \dot{p}_1 &= -k(x_2 - x_1 - L)\\
\dot{p}_2 &= -\frac{\partial H}{\partial x_2} &\Longrightarrow\quad \dot{p}_2 &= k(x_2 - x_1 - L) - k(x_3 - x_2 - L)\\
& & &= -k(2x_2 - x_1 - x_3)\\
\dot{p}_3 &= -\frac{\partial H}{\partial x_3} &\Longrightarrow\quad \dot{p}_3 &= k(x_3 - x_2 - L)
\end{aligned}$$

Question 2 (3)

- Momentum

$$p = p_1 + p_2 + p_3 = m_1\dot{x}_1 + m_2\dot{x}_2 + m_3\dot{x}_3$$

- Closed system $\dot{p} = 0$

Question 3

- Energies

$$KE = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \qquad PE = \frac{\mu}{2}\left(x_2 - x_1 - L\right)^2$$

- Lagrangian

$$L = KE - PE = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{\mu}{2}\left(x_2 - x_1 - L\right)^2$$

- Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1} \implies m_1\ddot{x}_1 = \mu\left(x_2 - x_1 - L\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = \frac{\partial L}{\partial x_2} \implies m_2\ddot{x}_2 = -\mu\left(x_2 - x_1 - L\right)$$

Question 4

- Energies

$$KE = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \qquad PE = -\frac{Gm_1m_2}{x_2 - x_1}$$

- Lagrangian

$$L = KE - PE = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{Gm_1m_2}{x_2 - x_1}$$

- Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1} \quad \Longrightarrow m_1\ddot{x}_1 = \frac{Gm_1m_2}{(x_2-x_1)^2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = \frac{\partial L}{\partial x_2} \quad \Longrightarrow m_2\ddot{x}_2 = -\frac{Gm_1m_2}{(x_2-x_1)^2}$$

Question 5 (1)

- Note $g = -9.8$
- Energies

$$
\begin{aligned}
KE &= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \\
PE &= \frac{1}{2}k\left(x_1 - L\right)^2 + \frac{1}{2}k\left(x_2 - x_1 - L\right)^2 \\
&\quad + \frac{1}{2}k\left(D - x_2 - L\right)^2 - m_1 g x_1 - m_2 g x_2
\end{aligned}
$$

- Hamiltonian

$$
\begin{aligned}
H &= KE + PE \\
&= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2}k\left(x_1 - L\right)^2 + \frac{1}{2}k\left(x_2 - x_1 - L\right)^2 \\
&\quad + \frac{1}{2}k\left(D - x_2 - L\right)^2 - m_1 g x_1 - m_2 g x_2
\end{aligned}
$$

Question 5 (2)

- Hamiltonian equations

$$
\begin{aligned}
\dot{x}_1 &= \frac{\partial H}{\partial p_1} &\Longrightarrow\quad& \dot{x}_1 = \frac{p_1}{m_1} \\
\dot{x}_2 &= \frac{\partial H}{\partial p_2} &\Longrightarrow\quad& \dot{x}_2 = \frac{p_2}{m_2} \\
\dot{p}_1 &= -\frac{\partial H}{\partial x_1} &\Longrightarrow\quad& \dot{p}_1 = -k\left(2x_1 - x_2\right) + mg \\
\dot{p}_2 &= -\frac{\partial H}{\partial x_2} &\Longrightarrow\quad& \dot{p}_2 = -k\left(2x_2 - x_1 - D\right) + mg
\end{aligned}
$$