Calculating velocity and acceleration

- Position given as a function of time

$$x(t) = f(t) \; , \qquad y(t) = g(t)$$

- Velocity: differentiate position with respect to time

$$v_x = x'(t) = f'(t) \; , \qquad v_y = y'(t) = g'(t)$$

- Acceleration: differentiate velocity with respect to time

$$a_x = x''(t) = f''(t) \; , \qquad a_y = y''(t) = g''(t)$$

Calculating velocity and position

- Acceleration given

$$a_x = f(t)\ , \qquad a_y = g(t)$$

- Velocity: integrate acceleration with respect to time

$$v_x = \int a_x dt = \int f(t)dt \qquad v_y = \int a_y dt = \int g(t)dt$$

- Position: integrate velocity with respect to time

$$x(t) = \int v_x(t)dt \qquad y(t) = \int v_y(t)dt$$

Special case

- Acceleration = gravity
- Initial velocity v_0 at an angle α

$$a_x = 0, \qquad a_y = -g$$

$$v_x = v_0 \cos\alpha \qquad v_y = -gt + v_0 \sin\alpha$$

$$x(t) = (v_0 \cos\alpha)\, t + x_0, \qquad y(t) = -\frac{gt^2}{2} + (v_0 \sin\alpha)\, t + y_0$$

Question 1

- Position

$$x(t) = t\cos t \qquad y(t) = -t\sin t$$

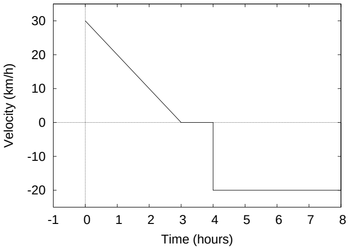
- Velocity

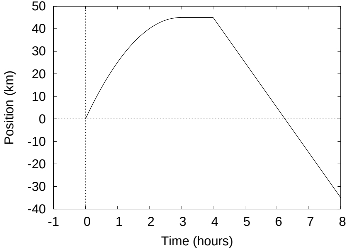
$$x'(t) = \cos t - t\sin t \qquad y'(t) = -\sin t - t\cos t$$

- Acceleration

$$x''(t) = -2\sin t - t\cos t \qquad y''(t) = -2\cos t + t\sin t$$

Question 2





- Cyclist ends up 35 km from starting point
- Average velocity =125 km /8 h=15.625km/h

Question 3

- Equations of position given by

$$x(t) = (v_0 \cos\alpha)\, t, \qquad y(t) = -\frac{gt^2}{2} + (v_0 \sin\alpha)\, t + H$$

- Substitute

$$t = \frac{x}{v_0 \cos\alpha} \Longrightarrow y(t) = -\frac{g}{2\,(v_0 \cos\alpha)^2} x^2 + x \tan\alpha + H$$

- Solve

$$-\frac{g}{2\,(v_0 \cos\alpha)^2} x^2 + x \tan\alpha + H = 0$$

- Solution (only choose the positive solution)

$$x_* = v_0 \cos\alpha \frac{v_0 \sin\alpha + \sqrt{v_0^2 \sin^2\alpha + 2gH}}{g}$$

- Same can be achieved with time

Question 4 (1)

- Calculate the positions

$$x(t) = (v_0 \cos\alpha)\, t, \qquad y(t) = -\frac{gt^2}{2} + (v_0 \sin\alpha)\, t$$

- Maximum height and distance

$$x_H = \frac{v_0 \cos\alpha}{g}\left(v_0 \sin\alpha + \sqrt{v_0^2 \sin^2\alpha - 2gH}\right)$$

$$y_{max} = \frac{v_0^2 \sin^2\alpha}{2g}$$

Question 4 (2)

- Introduce L: distance to the step
- Condition

$$y(L) > H$$

- Calculate the time and height

$$t = \frac{L}{v_0 \cos\alpha} \implies y(L) = -\frac{gL^2}{2v_0^2\cos^2\alpha} + L\tan\alpha$$

- Corresponding value of v_0:

$$y(L) \geq H \implies v_0 \geq \sqrt{\frac{gL^2}{2\cos^2\alpha\,(L\tan\alpha - H)}}$$

Question 5(1)

- Calculate the positions

$$x(t) = (v_0 \cos\alpha)\, t, \qquad y(t) = -\frac{gt^2}{2} + (v_0 \sin\alpha)\, t$$

- Condition

$$y(L) > 0$$

- Calculate the time and height

$$t = \frac{L}{v_0 \cos\alpha} \implies y(L) = -\frac{gL^2}{2v_0^2 \cos^2\alpha} + L\tan\alpha$$

- Corresponding value of v_0:

$$y(L) > 0 \implies v_0 > \sqrt{\frac{gL}{\sin(2\alpha)}}$$

Question 5(2)

- Maximum height

$$\frac{dy}{dt} = 0 \Longrightarrow t_{max} = \frac{v_0 \sin\alpha}{g} \Longrightarrow y_{max} = \frac{v_0^2 \sin^2\alpha}{2g}$$

- Stone touches the ground for $y(t) = -H$
- Corresponding time

$$-H = -\frac{gt^2}{2} + (v_0 \sin\alpha)\, t$$

- Solution (only choose the positive solution)

$$x_* = 2v_0 \cos\alpha \frac{v_0 \sin\alpha + \sqrt{v_0^2 \sin^2\alpha + 2gH}}{g}$$

Question 6 (1)

- Calculate the positions

$$x(t) = (v_0 \cos\alpha)\, t, \qquad y(t) = -\frac{gt^2}{2} + (v_0 \sin\alpha)\, t$$

- Angles 1 verify: $y(L) = 0$
- Angles 2 verify: $y(L + \Delta L) = 0$
- Calculate the time and height

$$t = \frac{L}{v_0 \cos\alpha} \implies y(L) = -\frac{gL^2}{2v_0^2 \cos^2\alpha} + L \tan\alpha$$

Question 6 (2)

- Corresponding values of α:

$$y(L) = 0 \implies \alpha_1 = \frac{1}{2}\sin^{-1}\left(\frac{gL}{v_0^2}\right) \; or \; \alpha_2 = \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{gL}{v_0^2}\right)$$

- Calculate the time and height for the second limit

$$t = \frac{L + \Delta L}{v_0 \cos\alpha} \implies y(L+\Delta L) = -\frac{g\,(L+\Delta L)^2}{2v_0^2\cos^2\alpha} + (L+\Delta L)\tan\alpha$$

- Corresponding values of α:

$$y\,(L+\Delta L) = 0 \implies \alpha_3 = \frac{1}{2}\sin^{-1}\left(\frac{g\,(L+\Delta L)}{v_0^2}\right)$$
$$or \quad \alpha_4 = \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{g\,(L+\Delta L)}{v_0^2}\right)$$

- Angle must verify $\alpha_1 \leq \alpha \leq \alpha_3$ or $\alpha_4 \leq \alpha \leq \alpha_2$