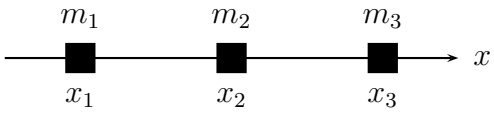
Lagrangian and Hamiltonian Mechanics

Question 1 *(2005 paper.)* Consider a one-dimensional system which consists of three particles of masses m_1, m_2, and m_3, with coordinates x_1, x_2, and x_3 $(x_1 \leq x_2 \leq x_3)$, interacting with one another through gravity.



(a) Using the one-dimensional version of Newton's Law of Gravity, determine the forces F_{12}, F_{21}, F_{13}, F_{31}, F_{23}, F_{32}, where F_{ij} is the force exerted by the j-th particle on the i-th particle.

(b) Using the relationship between the potential energy U of the system and the corresponding forces, show that the above expressions correspond to

$$U(x_1, x_2, x_3) = -\frac{\gamma m_1 m_2}{x_2 - x_1} - \frac{\gamma m_2 m_3}{x_3 - x_2} - \frac{\gamma m_3 m_1}{x_3 - x_1}.$$

(c) Write down the expression for the Hamiltonian H of the system.

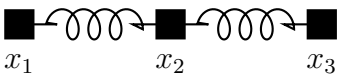
(d) Write down the Hamiltonian equations for this system.

(e) Write down the expression for the momentum P of this system.

(f) Show that P is an integral of motion (conserved quantity).

(g) Write down the expression for the Lagrangian L of the system and derive the Lagrangian form of the governing equations.

Question 2 (*2006 paper*.) Consider a one-dimensional system which consists of three particles of masses m_1, m_2, and m_3, with coordinates x_1, x_2, and x_3 $(x_1 \leq x_2 \leq x_3)$ connected by two identical springs of modulus μ and free length L:



(a) Write down the expression for the Hamiltonian H of this system.

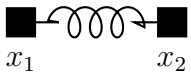
(b) Write down the Hamiltonian equations for this system.

(c) Write down the expression for the momentum P of this system.

(d) Prove that P is conserved.

(e) Write down the expression for the Lagrangian L of the system and derive the Lagrangian form of the governing equations.

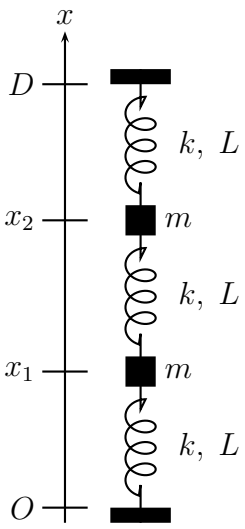
Question 3 (*2007 paper.*) Consider a one-dimensional system which consists of two particles of masses m_1 and m_2, with coordinates x_1 and x_2 $(x_1 < x_2)$ connected by a spring of modulus μ and free length L:



Write down the expression for the Lagrangian of the system, and derive the Lagrangian form of the governing equations.

Question 4 *(2008 paper.)* Consider a one-dimensional system which consists of two particles of masses m_1 and m_2, with coordinates x_1 and x_2 $(x_1 < x_2)$ interacting through gravity. Write down the expression for the Lagrangian of the system, and derive the Lagrangian form of the governing equations.

Question 5 (*2009 paper.*) Two identical particles of mass m are attached to three identical springs (modulus k and unperturbed length L) as shown of figure. The top and bottom springs are attached to fixed supports. The distance between the fixed supports is denoted by D.



We denote by x_i and $p_i = m_i \dot{x}_i$ the position and momentum of particle i $(i = 1, 2)$, respectively.

(a) Write down the expression for the total energy H of the system of this system in terms of x_i and p_i $(i = 1, 2)$. Note that the potential energy is equal to the sum of the gravitational potential energy of the two particles and of the potential energy of the 3 springs.

(b) Write down the 4 Hamiltonian equations of this system.