

# MS4414 Midterm

Answer all three questions

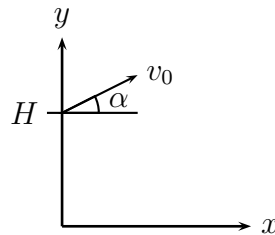
Thursday 10th March 2011

**Question 1.** The Cartesian coordinates of a particle on the plane are given by

$$x = (1 + t) \cos t, \quad y = 2(1 + t) \sin t, \quad t : 0 \rightarrow \pi.$$

Find the velocity and acceleration of the particle, and sketch its trajectory (show as much detail as possible).

**Question 2.** A stone is thrown with velocity  $v_0$ , at an angle  $\alpha$  to the horizontal, from height  $H$ , as shown in the diagram below.



Calculate: (a) the  $x$ -coordinate of the point where the stone hits the ground, (b) the speed at which the stone hits the ground.

**Question 3.** A sling  $L$  metres long, with a stone of mass,  $m$  kilograms, is being rotated with angular velocity  $\omega$  radians per second. Find the period of the rotations, the linear velocity and the acceleration of the stone.

## Theoretical Mechanics: Summary

### Kinematics

1. Position vector  $\mathbf{r}$ , velocity  $\mathbf{v}$ , and acceleration  $\mathbf{a}$  of a particle are related by:

$$\mathbf{v} = \dot{\mathbf{r}}, \quad \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}.$$

2. 1D motion with constant velocity  $v$ :

$$x = vt + x_0.$$

3. 1D motion with constant acceleration  $a$ :

$$v = at + v_0, \quad x = \frac{1}{2}at^2 + v_0t + x_0,$$

where  $x_0$  and  $v_0$  are the position and velocity at  $t = 0$ , respectively.

4. Rotation with constant angular velocity  $\omega$  (frequency  $\nu = \frac{\omega}{2\pi}$ ) along a circle of radius  $R$ :

- polar coordinates

$$r = R, \quad \theta = \omega t + \theta_0,$$

- Cartesian coordinates

$$x = R \cos(\omega t + \theta_0), \quad y = R \sin(\omega t + \theta_0),$$

- linear velocity

$$v = R\omega,$$

- acceleration

$$a = R\omega^2,$$

where  $\theta_0$  is the value of  $\theta$  at  $t = 0$ .

5. Rotation with constant angular acceleration  $\alpha$ :

$$\omega = \alpha t + \omega_0, \quad \theta = \frac{1}{2}\alpha t^2 + \omega_0 t + \theta_0,$$

where  $\omega$  is the angular velocity,  $\theta$  is the angular coordinate,  $\omega_0$  is the angular velocity at  $t = 0$ , and  $\theta_0$  is the angular coordinate at  $t = 0$ .

## Dynamics

1. Newton's second law:

$$m\mathbf{a} = \mathbf{F}.$$

2. For a sliding body, the friction force is  $F_{\text{fr}} = kN$ , when  $N$  is the normal reaction force. It is oriented in the opposite sense of the motion.

3. Conserved quantities:

- linear momentum

$$\mathbf{P} = \sum_i m_i \mathbf{v}_i,$$

- angular momentum with respect to the origin

$$\mathbf{A}_O = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i,$$

- angular momentum with respect to an arbitrary point  $P$

$$\mathbf{A}_P = \sum_i m_i (\mathbf{r}_i - \mathbf{r}_P) \times \dot{\mathbf{r}}_i,$$

- total energy

$$E = U(x_1, x_2, \dots) + \sum_i \frac{1}{2} m_i v_i^2,$$

where  $U$  is the potential energy.

4. A conservative force  $\mathbf{F}$  and the corresponding potential energy  $U$  are related by

$$\mathbf{F} = -\nabla U.$$

5. The potential energy and force for a spring of modulus  $k$ , and unperturbed length  $L_0$  are

$$U = \frac{1}{2} k (L - L_0)^2, \quad F = -k (L - L_0),$$

where  $L$  is the current length of the spring. The direction of  $\mathbf{F}$  is such that it tries to bring the spring back to its unperturbed configuration.

6. The potential energy  $U$  and force  $F$  for a particle of mass  $m$  located at a height  $H$ , in the Earth's gravitational field are

- locally:

$$U = -mgH, \quad \mathbf{F} = m\mathbf{g},$$

- globally:

$$U = -\frac{GM_E m}{R_E + H}, \quad \mathbf{F} = -\frac{GM_E m}{(R_E + H)^2} \mathbf{e}_r,$$

where  $G$  is the gravitational constant,  $M_E$  is the Earth's mass,  $R_E$  is the Earth's radius,  $\mathbf{e}_r$  is a radial unit vector.  $G = 6.7 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ ,  $M_E = 6.0 \times 10^{24} \text{ kg}$ , and  $R_E = 6.4 \times 10^6 \text{ m}$ .

- The angular velocity of a body rotating along a circular orbit around a much heavier body of mass  $M$  is

$$\omega = \sqrt{\frac{GM}{R^3}}$$

where  $R$  is the orbit radius.

## Oscillations

The equation of a forced linear pendulum with small amplitude is

$$\ddot{\phi} + 2c\dot{\phi} + \omega^2\phi = F_0 \cos(\Omega_0 t),$$

where  $c$  is the friction coefficient,  $\omega^2 = \frac{L}{g}$  is the natural frequency of the pendulum,  $L$  is the length of the pendulum,  $F_0$  and  $\Omega_0$  are the amplitude and frequency of the external forcing.

## Hamiltonian mechanics

1. The Hamiltonian equations are

$$\dot{x}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial x_j}, \quad 1 \leq j \leq n.$$

2. The Poisson brackets of functions

$$\{F, G\} = \sum_{i=1}^n \left( \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x_i} - \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial p_j} \right).$$

3. A transformation

$$x'_i = x'_i(x_1, \dots, x_n, p_1, \dots, p_n), \quad p'_i = p'_i(x_1, \dots, x_n, p_1, \dots, p_n),$$

is canonical if and only if

$$\{x'_i, p'_k\} = -\delta_{ik}, \quad \{x'_i, x'_k\} = 0, \quad \{p'_i, p'_k\} = 0.$$

4. The Lagrangian equations are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_j} \right) - \frac{\partial L}{\partial x_j} = 0, \quad 1 \leq j \leq n.$$

## Stability of dynamical systems

Let  $\mathbf{x}_F$  be a fixed point of a dynamical system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ , where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_1(x_1, \dots, x_k) \\ f_2(x_1, \dots, x_k) \\ \vdots \\ f_k(x_1, \dots, x_k) \end{bmatrix}.$$

Then,

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_k} \end{bmatrix},$$

is the Jacobian matrix of the system at  $\mathbf{x}_F$ , with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$ .

- If  $\Re(\lambda_j) < 0$  for all  $j$  then  $\mathbf{x}_F$  is asymptotically stable.
- If  $\Re(\lambda_j) > 0$  for some  $j$  then  $\mathbf{x}_F$  is unstable.
- If  $\Re(\lambda_j) < 0$  for some  $j$ , and  $\Re(\lambda_j) = 0$  for the remaining  $j$ , then the test is inconclusive.