Lagrangian Mechanics

MS4414 Theoretical Mechanics

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1 Introduction

Newton's equations describe all mechanical systems. However sometimes it is convenient to use an alternative formulation using the Lagrangian or Hamiltonian equations of motion. These formulations are useful when the systems are best described by unusual coordinate systems.

A simple example is a pendulum which is best described by equations of motion for the pendulum angle θ.

N.B. Neither Lagrangian nor Hamiltonian mechanics can handle friction.

2 The Principle of Least Action

Action The action of a system S can be calculated from the Lagrangian L:

$$S_{\text{AB}} = \int_A^B L(t)\ \mathrm{d}t \tag{1}$$

The Lagrangian is given by

$$L = K - V \tag{2}$$

where K is kinetic energy and V is the potential energy

Lagrangian Equations of Motion Lagrange's equations of motion are derived by finding a trajectory $x(t)$ which minimises the action

$$S = \int_A^B L(t, x, \dot{x})\ \mathrm{d}t \tag{3}$$

If $x(t)$ is the minimal trajectory then S is unchanged if $x(t) \to x(t) + \delta x(t)$ where $\delta x(t_A) = \delta x(t_B) = 0$.

$$S + \delta S = \int_A^B L(t, x + \delta x, \dot{x} + \delta \dot{x})\ \mathrm{d}t \tag{4}$$

where $\delta \dot{x} = \frac{\mathrm{d}\,\delta x}{\mathrm{d}t}$

$$S + \delta S = \int_A^B \left[L(t, x, \dot{x}) + \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} \right]\ \mathrm{d}t \tag{5}$$

Subtracting S from either side

$$\delta S = \int_A^B \left[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} \right]\ \mathrm{d}t \tag{6}$$

Rewriting the second term

$$\delta S = \int_A^B \left[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}} \frac{\mathrm{d}\,\delta x}{\mathrm{d}t} \right]\ \mathrm{d}t \tag{7}$$

If $x(t)$ is the particle trajectory, which minimises S, then $\delta S = 0$ i.e.

$$0 = \int_A^B \left[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}} \frac{\mathrm{d}\,\delta x}{\mathrm{d}t} \right]\ \mathrm{d}t \tag{8}$$

Integrate the second term by parts (boundary term is zero because $\delta x(t_A) = \delta x(B) = 0$)

$$0 = \int_A^B \left[\frac{\partial L}{\partial x}\delta x - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}}\delta x \right]\ \mathrm{d}t \tag{9}$$

$$0 = \int_A^B \left[\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}} \right] \delta x\, \mathrm{d}t \tag{10}$$

This integral must be zero for any (small) choice of $\delta x(t)$. The only way this can be true is if the term in square brackets is zero. Thus Lagrange's equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{11}$$

Remember that the Lagrangian is given by $L = T - V$ where T is kinetic energy, and V is potential energy.

In the case where a system is described not by a single coordinate x but by a collection of coordinates q_i, $i = 1 \ldots n$, there are n Lagrangian equations of motion each of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \tag{12}$$

The coordinates q_i do *not* have to be the Cartesian coordinates of the particles making up the system. They can be any sort of coordinate, e.g. the angle of a pendulum, the distance a particle travels along a magnetic field line. This is the power of the Lagrangian and Hamiltonian approaches.

3 Examples

Some examples to show that Lagrangian mechanics and Newtonian mechanics are equivalent.

Free Particle According the Newton's first law a particle on which no forces are acting will move at constant velocity.

$$L(\dot{x}, x, t) = \frac{m}{2}\dot{x}^2$$

Lagrangian equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

Substituting in the Lagrangian for a free particle

$$m\ddot{x} = 0$$

Integrate

$$\dot{x} = \text{const.}$$

in agreement with Newton's second law.

Particle in a conservative force field A particle of mass, m moves in a conservative force-field with potential energy $V(\mathbf{x})$. Newton's second law states that the

$$m\ddot{\mathbf{x}} = -\boldsymbol{\nabla} V$$

The Lagrangian of the system

$$L(\dot{\mathbf{x}}, \mathbf{x}) = \frac{m}{2}\dot{\mathbf{x}}^2 - V(x)$$

Lagrangian equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i}, \qquad i = 1, 2, 3$$

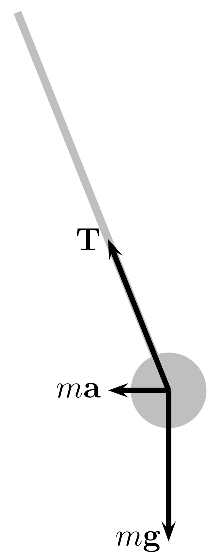
substitute in the Lagrangian

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i}$$

write in vector form

$$m\ddot{\mathbf{x}} = -\boldsymbol{\nabla} V$$

Pendulum The equations of motion of pendulum with a large angle of oscillation are hard to derive from Newton's Laws. They are simple (simpler!) to derive in the Lagrangian framework. If the length of the pendulum is r and the angle is θ ($\theta = 0$ is the ground state of the pendulum)



$$T = \frac{1}{2} m r^2 \dot{\theta}^2 \tag{13}$$

$$V = mgr\left(1 - \cos\theta\right) \tag{14}$$

$$L = \frac{1}{2} m r^2 \dot{\theta}^2 - mgr\left(1 - \cos\theta\right) \tag{15}$$

$$\frac{\partial L}{\partial \theta} = -mgr\sin\theta \tag{16}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = mr^2\ddot{\theta} \tag{18}$$

Thus the Lagrangian equations of motion are

$$mr^2\ddot{\theta} = -mgr\sin\theta \tag{19}$$

As mentioned above, Lagrangian equations of motion can only be derived for systems without friction (otherwise a potential energy cannot be defined). If I want to derive equations for a pendulum with friction in the Lagrangian framework I have to derive the friction free equations and then return to the Newtonian framework to add in friction terms by hand.

$$mr^2\ddot{\theta} = -mgr\sin\theta - \gamma mr\dot{\theta} \tag{20}$$

Exam Question (2008r) Consider a one-dimensional system which consists of two particles of masses m_1 and m_2, with coordinates x_1 and x_2 $(x_1 < x_2)$ interacting through gravity. Write down the expression for the Lagrangian of the system and derive the Lagrangian form of the governing equations.

Kinetic energy: $K = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2$

Potential energy: $V = -\frac{Gm_1m_2}{x_2 - x_1}$

Lagrangian: $L = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 + \frac{Gm_1m_2}{x_2 - x_1}$

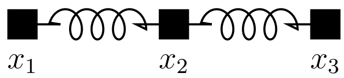
Lagrangian equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}, \qquad i = 1, 2.$$

Particle 1: $m_1\ddot{x}_1 = \dfrac{Gm_1m_2}{(x_2 - x_1)^2}$

Particle 2: $m_2\ddot{x}_2 = -\dfrac{Gm_1m_2}{(x_2 - x_1)^2}$

Exam Question 2007r Consider a one-dimensional system which consists of three particles of masses m_1, m_2, and m_3, with coordinates x_1, x_2, and x_3 $(x_1 \leq x_2 \leq x_3)$ connected by two identical springs of modulus μ and free length L:



Write down the expression for the Lagrangian L of the system, and derive the Lagrangian form of the governing equations.

Kinetic energy: $K = \dfrac{m_1}{2}\dot{x}_1^2 + \dfrac{m_2}{2}\dot{x}_2^2 + \dfrac{m_3}{2}\dot{x}_3^2$

Potential energy: $V = \dfrac{\mu}{2}(x_3 - x_2 - L)^2 + \dfrac{\mu}{2}(x_2 - x_1 - L)^2$

Lagrangian:

$$L = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 + \frac{m_3}{2}\dot{x}_3^2 - \frac{\mu}{2}(x_3 - x_2 - L)^2 - \frac{\mu}{2}(x_2 - x_1 - L)^2$$

Lagrangian equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}, \qquad i = 1, 2, 3.$$

$$m_1\ddot{x}_1 = \mu(x_3 - x_2 - L)$$
$$m_2\ddot{x}_2 = -\mu(x_3 - x_2 - L) + \mu(x_2 - x_1 - L)$$
$$m_3\ddot{x}_3 = -\mu(x_2 - x_1 - L)$$

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4 Conserved Quantities

4.1 Conservation of Momentum

If there a collection of particles only interact with each other

$$0 = \sum_\alpha \frac{\partial L}{\partial x_{i,\alpha}}$$

Using the Lagrangian equations of motion

$$0 = \sum_\alpha \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_{i,\alpha}}\right)$$

Integrate

$$\text{const} = \sum_\alpha \frac{\partial L}{\partial \dot{x}_{i,\alpha}}$$

For standard systems

$$L = \sum_\alpha \frac{m}{2}\dot{\mathbf{x}}_\alpha^2 - V(\mathbf{x}_\alpha)$$

$$\text{const} = \sum_\alpha \frac{\partial L}{\partial \dot{x}_{i,\alpha}} = \sum_\alpha m_\alpha \dot{x}_{i,\alpha}$$

Or

$$\mathbf{const} = \sum_\alpha m_\alpha \dot{\mathbf{x}}_\alpha$$

i.e. conservation of momentum.

4.2 Conservation of Angular Momentum

The Lagrangian should be unchanged when positions $x_{i,\alpha}$ and velocities are rotated through an angle $\delta\phi$:

$$\mathbf{x}_\alpha \to \mathbf{x}_\alpha + \delta\phi \times \mathbf{x}, \qquad \dot{\mathbf{x}}_\alpha \to \dot{\mathbf{x}}_\alpha + \delta\phi \times \dot{\mathbf{x}}$$

$$0 = \frac{\partial L}{\partial \mathbf{x}_\alpha} \cdot \delta\phi \times \ \mathbf{x}_\alpha + \frac{\partial L}{\partial \dot{\mathbf{x}}_\alpha} \cdot \delta\phi \times \dot{\mathbf{x}}_\alpha$$

Cycle the triple product

$$0 = \delta\phi \cdot \sum_{\alpha} \left[\mathbf{x}_\alpha \times \frac{\partial L}{\partial \mathbf{x}_\alpha} + \dot{\mathbf{x}}_\alpha \times \frac{\partial L}{\partial \dot{\mathbf{x}}_\alpha} \right]$$

Apply the Lagrangian equations of motion

$$0 = \delta\phi \cdot \sum_{\alpha} \left[\mathbf{x}_\alpha \times \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{\mathbf{x}}_\alpha} \right) + \dot{\mathbf{x}}_\alpha \times \frac{\partial L}{\partial \dot{\mathbf{x}}_\alpha} \right]$$

Valid for arbitrary $\delta\boldsymbol{\phi}$

$$\mathbf{0} = \sum_{\alpha} \left[\mathbf{x}_\alpha \times \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{\mathbf{x}}_\alpha} \right) + \dot{\mathbf{x}}_\alpha \times \frac{\partial L}{\partial \dot{\mathbf{x}}_\alpha} \right]$$

$$\frac{\partial L}{\partial \dot{\mathbf{x}}_\alpha} = m_\alpha \dot{\mathbf{x}}_\alpha$$

$$\mathbf{0} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{\alpha} \left[\mathbf{x}_\alpha \times \frac{\partial L}{\partial \dot{\mathbf{x}}_\alpha} \right]$$

$$\mathbf{0} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{\alpha} \left[\mathbf{x}_\alpha \times (m_i \dot{\mathbf{x}}_\alpha) \right]$$

I.e. conservation of angular momentum.