Kinematics Examples

MS4414 Theoretical Mechanics

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1 Neglecting Air Resistance [1]

The criteria for neglecting air resistance in calculating the trajectory of a spherical object of denity ρ, diameter D, launched with velocity v_0 is

$$\frac{3\rho_{\mathrm{a}} C_{\mathrm{D}} v_0}{4\rho g D} \ll 1$$

where $\rho_a = 1.24\,\mathrm{kg\,m^{-3}}$ is the density of air, $C_{\mathrm{D}} = 0.3$ is the drag coefficient. To understand why this is so, you will have to wait for courses on fluid mechanics and mathematical modelling.

Quantity	Shot putt	Basketball	Baseball	Tennis ball	Golf ball
D (m)	0.11	0.243	0.073	0.064	0.042
ρ $(10^3\,\mathrm{kg\,m^{-2}})$	10.4	0.08	0.699	0.423	1.17
v_0 $(\mathrm{m\,s^{-1}})$	13.7	7.62	42.7	45.7	70.1
$\frac{3\rho_{\mathrm{a}} C_{\mathrm{D}} v_0}{4\rho g D}$	0.005	0.085	1.01	2.19	2.83

2 Shot Putting [1]

A shot putter releases a shot from a height of $2.3\,\mathrm{m}$ at a speed of $13.7\,\mathrm{m\,s^{-1}}$ and at an angle of $41.9°$ (this angle maximises the horizontal distance the shot travels.

1. Derive the time dependent equations of the trajectory in parametric form: $x = x(t)$ and $y = y(t)$.
2. Calculate the equation of the trajectory in the form $y = y(x)$
3. Find the location x_{max}, y_{max} of the maximum in the trajectory.
4. Find the time, T, the shot is airborne.
5. Find the horizontal distance, L, travelled by the shot.
6. Verify L is greater than L_{45}, the distance the shot would have travelled if the launch angle was $45°$.

Equations of the trajectory: $\boldsymbol{x(t), y(t)}$ The equations

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$

Integrate once

$$\int_{v_0\cos\theta}^{v_x(t)} \mathrm{d}\dot{x} = \int_0^t 0\,\mathrm{d}t \qquad \int_{v_0\sin\theta}^{v_y(t)} \mathrm{d}\dot{y} = -\int_0^t g\,\mathrm{d}t$$

with the result

$$v_x(t) = v_0\cos\theta \qquad v_y(t) = v_0\sin\theta - gt$$

Integrate a second time

$$\int_0^{x(t)} \mathrm{d}x = \int_0^t v_0\cos\theta \qquad \int_H^{y(t)} \mathrm{d}y = \int_0^t (v_0\sin\theta - gt)\,\mathrm{d}t$$

with the result

$$x(t) = v_0 t\cos\theta \qquad y(t) = H + v_0 t\sin\theta - \tfrac{1}{2}gt^2$$

Equations of the trajectory: $\boldsymbol{y(x)}$ Eliminate time from the two equations

$$t = \frac{x}{v_0\cos\theta} \implies y(x) = H + x\tan\theta - \frac{gx^2}{2v_0^2\cos^2\theta}$$

Maximum of trajectory

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta - \frac{gx}{v_0^2\cos^2\theta}$$

$x = x_{\text{max}}$ when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$$x_{\text{max}} = \frac{v_0^2\sin\theta\cos\theta}{g}$$

Substituting in numerical values

$$x_{\text{max}} = \frac{13.7^2\sin 41.9^\circ\cos 41.9^\circ}{9.81} = 9.51\,\text{m}$$

To obtain y_{max} substitute x_{max} into the equation of the trajectory

$$y_{\text{max}} = H + \frac{v_0^2\sin^2\theta}{2g}$$

$$y_{\text{max}} = 2.3 + \frac{13.7^2\sin^2 41.9^\circ}{2\times 9.81} = 6.57$$

Airborne Time Find the time T at which $y = 0$

$$0 = H + v_0 T\sin\theta - \frac{g}{2}T^2 \implies T^2 - \frac{2v_0\sin\theta}{g}T - \frac{2H}{g}$$

Quadratic formula (positive root)

$$T = \frac{v_0\sin\theta}{g} + \sqrt{\frac{v_0^2\sin^2\theta}{g^2} + \frac{2H}{g}}$$

Substitute values in

$$T = \frac{13.7\sin 41.9}{9.81} + \sqrt{\frac{13.7^2\sin^2 41.9}{9.81^2} + \frac{2\times 2.3}{9.81}} = 2.09\,\text{s}$$

Distance travelled We can calculate L by using $L = x(T)$

$$L = v_0 T\cos\theta.$$

The substitution is best done numerically

$$L = 13.7\times 2.09\times\cos 41.9 = 21.3\,\text{m}$$

45° case Find numerical values of T_{45} and L_{45} using the formulae derived above. Substitute $\theta = 45°$ into:

$$T_{45} = \frac{v_0 \sin\theta}{g} + \sqrt{\frac{v_0^2 \sin^2\theta}{g^2} + \frac{2H}{g}}$$

to obtain

$$T_{45} = \frac{13.7 \sin 45}{9.81} + \sqrt{\frac{13.7^2 \sin^2 45}{9.81^2} + \frac{2 \times 2.3}{9.81}} = 2.19\,\mathrm{s}$$

And substitute $L = L_{45}$, $T = T_{45}$ and $\theta = 45°$ into

$$L = v_0 T \cos\theta$$

to obtain

$$L_{45} = 13.7 \times 2.19 \times \cos 45 = 21.2\,\mathrm{m} < L = 21.3\,\mathrm{m}$$

3 Throwing a Basketball [1]

A basketball player is a horizontal distance L from the hoop. He throws the basketball with velocity v_0 from height H to a hoop at height h.

1. Derive a formula for the trajectory of the basketball in the form $x = x(t)$, $y = y(t)$
2. Derive a formula for the trajectory of the basketball in the form $y = y(x)$.
3. Derive a formula for the two possible launch angles of the ball.
4. Taking the values $v_0 = 7.62\,\mathrm{m\,s^{-1}}$, $L = 4.57\,\mathrm{m}$, $H = 2.13\,\mathrm{m}$ and $h = 3.05\,\mathrm{m}$ calculate the values of those two angles.
5. Given that the diameter of the basketball is $0.244\,\mathrm{m}$ and the diameter of the hoop is $0.45\,\mathrm{m}$ show that one trajectory is impractical.
6. Show that the basketball achieves a maximum height of $4.33\,\mathrm{m}$
7. Show that it takes the basketball $1.2\,\mathrm{s}$ to reach the hoop.

Trajectory: $x(t), y(t)$ The equations of motion are:

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$

Integrate once

$$\int_{v_0 \cos\theta}^{v_x(t)} \mathrm{d}\dot{x} = \int_0^t 0\,\mathrm{d}t \qquad \int_{v_0 \sin\theta}^{v_y(t)} \mathrm{d}\dot{y} = -\int_0^t g\,\mathrm{d}t$$

with the result

$$v_x(t) = v_0 \cos\theta \qquad v_y(t) = v_0 \sin\theta - gt$$

Integrate a second time

$$\int_0^{x(t)} \mathrm{d}x = \int_0^t v_0 \cos\theta \qquad \int_H^{y(t)} \mathrm{d}y = \int_0^t (v_0 \sin\theta - gt)\,\mathrm{d}t$$

with the result

$$x(t) = v_0 t \cos\theta \qquad y(t) = H + v_0 t \sin\theta - \tfrac{1}{2} g t^2$$

Trajectory $y = y(x)$ Eliminate time from the two equations

$$t = \frac{x}{v_0 \cos\theta} \implies y(x) = H + x \tan\theta - \frac{g x^2}{2 v_0^2 \cos^2\theta}$$

Analytic throw angles Given that the location of the hoop $y = h$, $x = L$ is a solution to the trajectory equation we have

$$h = H + L \tan\theta - \frac{g L^2}{2 v_0^2 \cos^2\theta}$$

To make this easier to solve we substitute in the identity[1]

$$\frac{1}{\cos^2\theta} = \tan^2\theta + 1$$

to obtain

$$h = H + L \tan\theta - \frac{g L^2}{2 v_0^2} - \frac{g L^2}{2 v_0^2} \tan^2\theta$$

rearrange

$$0 = \tan^2\theta - \frac{2 v_0^2}{g L} \tan\theta + 1 + \frac{2\,(h - H)\,v_0^2}{g L^2}$$

[1]Divide $1 = \cos^2\theta + \sin^2\theta$ by $cos^2\theta$.

and solve for $\tan\theta$ with the quadratic formula

$$\tan\theta = \frac{v_0^2}{gL} \pm \sqrt{\frac{v_0^4}{g^2L^2} - 1 - \frac{2\,(h-H)\,v_0^2}{gL^2}}$$

or

$$\tan\theta = \frac{v_0^2}{gL}\left[1 \pm \sqrt{1 - \frac{gL}{v_0^2}\left(\frac{gL}{v_0^2} + \frac{2\,(h-H)}{L}\right)}\right]$$

Numerical values of angles

$$\tan\theta = \frac{7.62^2}{9.81\times 4.57}\left[1 \pm \sqrt{1 - \frac{9.81\times 4.57}{7.62^2}\left(\frac{9.81\times 4.57}{7.62^2} + \frac{2\,(3.05-2.13)}{4.47}\right)}\right]$$

$$\theta = 59.2^\circ,\ 42.5^\circ$$

Entering the hoop Calculate the gradient of the trajectory when the ball reaches the hoop

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta - \frac{gL}{v_0^2\cos^2\theta}$$

In order to enter the hoop the absolute value of the trajectory's gradient must be greater than $d_\mathrm{B}/d_\mathrm{H} = 0.244/0.45 = 0.542$.

$$\theta = 42.5^\circ \qquad \left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| = 0.475$$

$$\theta = 59.2^\circ \qquad \left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| = 1.2$$

Only if $\theta = 59.2^\circ$ will the ball enter the hoop.

Maximum height Differentiate

$$y = H + x\tan\theta - \frac{gx^2}{2v_0^2\cos^2\theta}$$

to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta - \frac{gx}{v_0^2\cos^2\theta}$$

at the maximum

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$x_{\text{max}} = \frac{v_0^2 \sin\theta \cos\theta}{g}$$

substitute into the equation for y

$$y_{\text{max}} = H + \frac{v_0^2 \sin^2\theta}{2g}$$

Substitute in numerical values

$$y_{\text{max}} = 2.13 + \frac{7.62^2 \sin^2 59.2}{2 \times 9.81} = 4.31\,\text{m}$$

Airborne time Use the equation $x = v_0 t \cos\theta$

$$L = v_0 T \cos\theta \implies T = \frac{L}{v_0 \cos\theta}$$

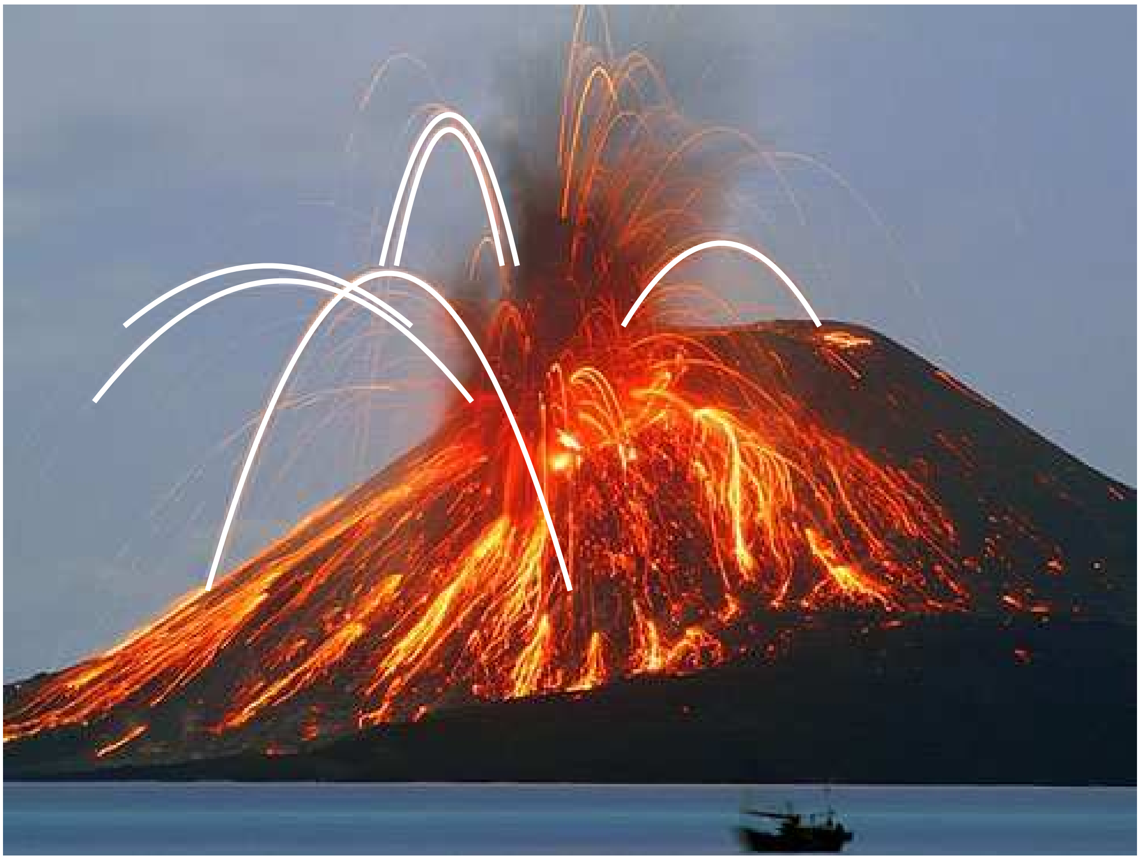
Substitute in numerical values

$$T = \frac{4.57}{7.62 \times \cos 59.2} = 1.17\,\text{s}$$

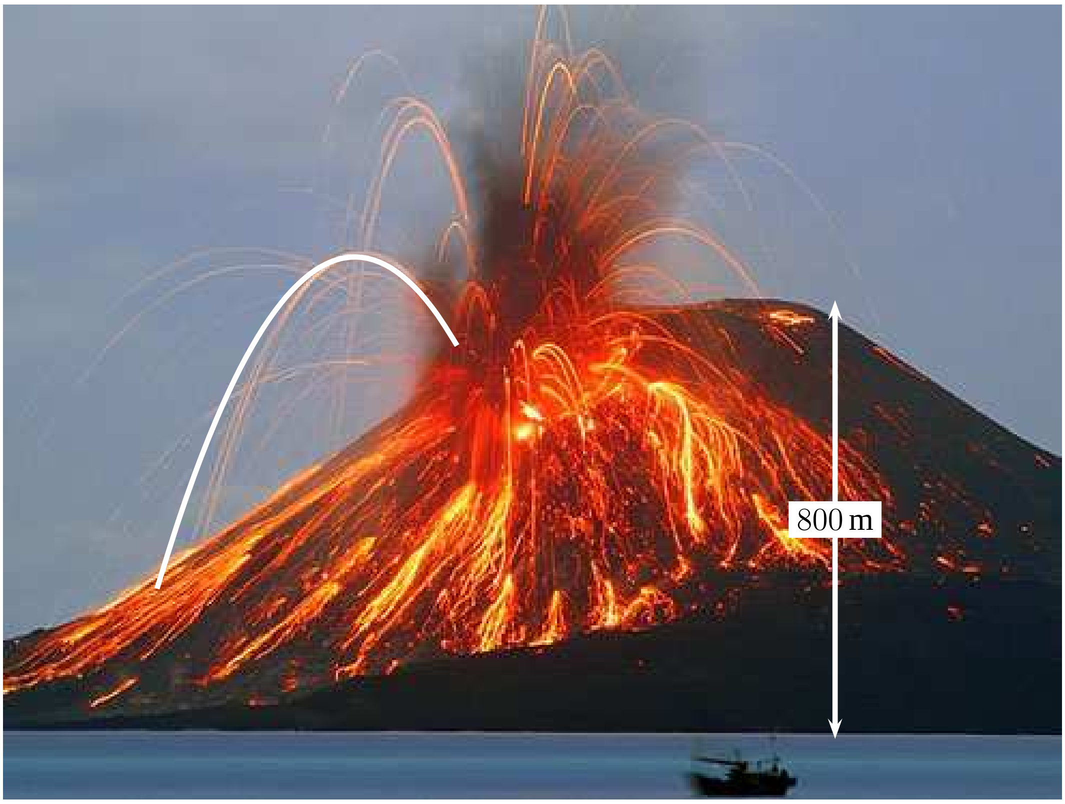
4 Volcanic Bombs

The figure [2] shows the type of eruption known as a fire fountain. It is a long exposure photo taken at night. The red curves are 'volcanic bombs' red hot lumps of magma ejected at high velocity from the volcano. Assuming the volcano is $800\,\mathrm{m}$ high estimate the velocity of the volcanic bombs.

The first question we ask is whether the trajectory of a volcanic bomb can be modelled by the equations we have derived. It would be a bit worrying if they couldn't but maybe air resistance is important. Use a graphics program to draw parabolas over some of the trajectories.



The parabolas fit the trajectories extremely well.



- 5.8 cm on the picture corresponds to 800 m
- Plot lines corresponding to $y = x\tan\theta - \frac{gx^2}{2v_0^2\cos^2\theta}$ for various values of v_0 and θ until a reasonable match to a trajectory is obtained.
- The line of the figure corresponds to $\theta = 60^\circ$ and $v_0 = 65\,\mathrm{m\,s^{-1}}$.
- If I was in less of a hurry I would use a digitising program like datatheif to get coordinates of points on the trajectory and then use a least squares fit to find best fit values for θ and v_0.
- Note that we only have an estimate of v_x and v_z; we don't know what v_y is up to.

The (octave) code I used to calculate the (scaled) trajectory

```
scale=800/5.8;
x=linspace(0,200,20);
g=9.81;
v0=65;
theta=60*pi/180;
y=x*tan(theta)-g*x.^2/2/v0^2/cos(theta)^2;
fprintf("\\pscurve");
for i=1:length(x);
  fprintf("(%f,%f)",6-x(i)/scale,6+y(i)/scale);
endfor;
fprintf("\n")
```

References

[1] Towing icebergs, falling dominos and other adventures in applied mathematics. Robert B. Banks

[2] http://www.skedaddel.com/blog/?tag=krakatoa