Hamiltonian Mechanics

MS4414 Theoretical Mechanics

William Lee

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1 Introduction

An alternative to the Lagrangian approach is to derive Hamilton's equations of motion. These are useful if one wants to solve a system numerically since unlike the Newtonian and Lagrangian approach to Hamiltonian approach gives first order differential equations which computers prefer to solve. The Hamiltonian of a system is a useful quantity in Quantum Mechanics.

2 The Hamiltonian

The Hamiltonian of a system is given by

$$H(p, q) = \min_{\dot{q}} [p\dot{q} - L(q, \dot{q})]$$

For example if the Lagrangian of the system is $L = \frac{1}{2}m\dot{q}^2 - V(q)$ then the Hamiltonian is given by $H = \frac{p^2}{2m} + V(q)$

To find the Hamiltonian equations of motion we take the partial derivatives of H with respect to p and q, remembering that $\dot{q}$ is no longer an independent quantity but is now a function of (in general) p and q.

$$\frac{\partial H}{\partial p} = \dot{q} + p\frac{\partial \dot{q}}{\partial p} - \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial p}$$

But in order for H to be a minimum with respect to $\dot{q}$ we must have $p = \frac{\partial L}{\partial \dot{q}}$. Thus the first Hamiltonian equation is

$$\dot{q} = \frac{\partial H}{\partial p}$$

Next we differentiate H with respect to q

$$\frac{\partial H}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \frac{\partial L}{\partial q} - \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q}$$

Since $p = \frac{\partial L}{\partial \dot{q}}$ the first and third terms cancel.

$$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}$$

I can use the Lagrangian equations of motion to replace $\frac{\partial L}{\partial q}$

$$\frac{\partial H}{\partial q} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}$$

And again using $p = \frac{\partial L}{\partial \dot{q}}$ gives us the second Hamiltonian equation

$$\dot{p} = -\frac{\partial H}{\partial q}$$

To summarise, the Hamiltonian equations of motion are

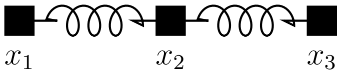
$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

For example if the Hamiltonian of the system is given by $H = \frac{p^2}{2m} + V(q)$ then

$$\dot{q} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial V}{\partial q}$$

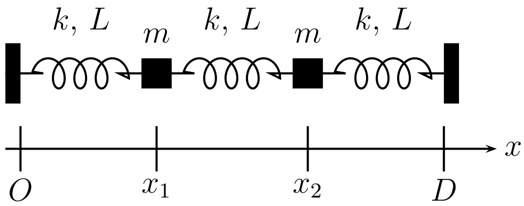
Both these equations should be familiar from Newtonian mechanics.

Exam Question 2007r Consider a one-dimensional system which consists of three particles of masses m_1, m_2 and m_3, with coordinates x_1, x_2 and x_3 ($x_1 \leq x_2 \leq x_3$) connected by two identical springs of modulus μ and free length L.



1. Write down the expression for the Hamiltonian, H, of this system
2. Write down the Hamiltonian equations for this system
3. Write down the expression for the momentum P of this system
4. Prove the P is conserved

Exam question 2009r Two identical particles of mass m are attached to three identical springs (modulus k and unperturbed length L) as shown in the figure. The left and right springs are attached to fixed walls. The distance between the fixed walls is denoted by D.



We suppose the only forces affecting the particles are the spring tensions. We do not consider the gravity force!

We denote by x_i and $p_i = m_i \dot{x}_i$ the position and momentum of particle i ($i = 1, 2$), respectively.

1. Write down the expression for the total energy H of the system in terms of x_i and p_i ($i = 1, 2$). Note that the potential energy is equal to the potential energy of the 3 springs.
2. Write down the 4 Hamiltonian equations of this system.