Dynamical Systems

MS4414 Theoretical Mechanics

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1 Eigenvalues and Eigenvectors Recap

Consider the matrix equations

$$\mathbf{M}\mathbf{v} = \lambda \mathbf{v}$$

This equation states that the action of the matrix $\mathbf{M}$ on a vector $\mathbf{v}$ is to multiply it by a constant λ. (In general the action of a matrix on a vector is to rotate and scale it.) These special vectors are called eignevectors $\mathbf{v}$ and the scaling factors λ.

If M is a symmetric $n \times n$ (Newton's third law ensures this that matrices describing mechanical systems are symmetric) then there are n such eigenvalues and their corresponding eigenvectors.

$$\mathbf{M}\mathbf{v}^{(i)} = \lambda_i \mathbf{v}^{(i)}$$

The eigenvectors are orthogonal and may be scaled to be orthonormal $\mathbf{v}^{(i)} \cdot \mathbf{v}^{(j)} = \delta_{ij}$

2 Stability

A general set of time evolution equations can be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

Fixed points are points for which $\dot{\mathbf{x}} = \mathbf{0}$. These points can be **stable** or **unstable**. A stable fixed point $\mathbf{x}_0$ is one for which points close to $\mathbf{x}_0$ move closer.

Consider the evolution of the point $\mathbf{x}_0 + \delta\mathbf{x}$ if $\delta\mathbf{x}$ is small

$$\delta\dot{\mathbf{x}} = \mathbf{J} \cdot \delta\mathbf{x}$$

where

$$J_{ij} = \left(\frac{\partial f_i}{\partial x_j}\right)_{\mathbf{x}_0}$$

Transform into the eigenvector system

$$\delta\dot{\mathbf{y}} = \mathbf{K} \cdot \delta\mathbf{y}$$

$\mathbf{K}$ is a diagonal matrix whose entries are the eigenvalues of the matrix $\mathbf{K}$:
$K_{ii} = \lambda_i$.

Solving the differential equations

$$\delta y_i(t) = \exp(\lambda_i t)\, \delta y_i(0)$$

If all the λ_i's are negative then the δy_i's decrease with time and the system is stable. If any of the λ_i's is positive then the corresponding δy_i will increase and thus the system is unstable (at that point).

To summarise: a dynamical system is described by the vector equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. A fixed point $\mathbf{x}_0$ is defined by $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$.

The fixed point is stable if the eigenvalues of the matrix

$$J_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x}_0}$$

are *all* negative.

The fixed point is unstable if *any* eigenvalue is positive.

If some eigenvalues are negative and some are zero then the results are inconclusive.

Exam Question 2007r Find and examine the fixed points of

$$\dot{\phi} = -\psi \qquad \dot{\psi} = \phi^2 - \psi\phi - 1$$

Find solutions to the equations $\dot{\phi} = \dot{\psi} = 0$. These are

$$\psi = 0, \qquad \phi = \pm 1$$

I.e.

$$\mathbf{x} = \begin{bmatrix} \phi \\ \psi \end{bmatrix}$$

Fixed points are $\mathbf{x}_1$ and $\mathbf{x}_2$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Matrix

$$\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 2\phi - \psi & -\phi \end{bmatrix}$$

First fixed point

$$\mathbf{J}_1 = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}, \qquad \lambda_1 = -\tfrac{1}{2} + i\tfrac{\sqrt{7}}{2}, \qquad \lambda_2 = -\tfrac{1}{2} - i\tfrac{\sqrt{7}}{2},$$

stable.

Second fixed point

$$\mathbf{J}_2 = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}, \qquad \lambda_1 = -1, \qquad \lambda = 2,$$

unstable.
