Conservation of Energy

MS4414 Theoretical Mechanics

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1 Conservative Force Fields

A large number of force fields $\mathbf{F}(x, y, z) = \mathbf{F}(\mathbf{r})$ can be written as the gradient of a potential energy $\phi(x, y, z)$

$$\mathbf{F} = -\nabla\phi$$

Where the notation above really means

$$\mathbf{F} = -\left(\boxed{}, \boxed{}, \boxed{}\right)$$

We will show that ϕ behaves like a potential energy.

1.1 One Dimension

In one dimension $F = -\frac{\mathrm{d}\phi}{\mathrm{d}x}$.

Consider Newton's second law $F = ma$, in terms of the displacement of the particle x the equation of motion is:

$$-\frac{\mathrm{d}\phi}{\mathrm{d}x} = \boxed{\qquad\qquad}$$

As before multiply both sides by the velocity of the particle

$$-\frac{\mathrm{d}\phi}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = m\boxed{\qquad\quad}\boxed{\qquad}$$

Write each side as a derivative

$$-\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\boxed{\qquad\qquad\qquad\qquad}\right]$$

Or, writing $\frac{\mathrm{d}x}{\mathrm{d}t} = v$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m}{2}v^2 + \phi\right]$$

Or integrating, constant of integration is the energy of the system

$$E = \frac{m}{2}v^2 + \phi$$

This form of the equation makes it clear that ϕ is the potential energy of the system.

Example: Gravity Newton's law of gravitation states that gravitational force on a mass m in the gravitational field of a body of mass M a distance r away is given by

$$F = -\frac{GMm}{\boxed{}}$$

We can interpret this as the derivative of a gravitational potential energy ϕ.

$$F = -\frac{GMm}{r^2} = -\frac{\mathrm{d}\phi}{\mathrm{d}r}$$

i.e.

$$\frac{GMm}{r^2} = \frac{\mathrm{d}\phi}{\mathrm{d}r}$$

Use an indefinite integral

$$\phi = \boxed{\phantom{\int \frac{GMm}{r^2}\,\mathrm{d}r}} + \phi_0$$

where ϕ_0 is a constant of integration which has no effect on the forcefield. It is traditional to take the zero of gravitational potential energy as when the bodies are infinitely separated: i.e. $\phi_0 = 0$.

$$\phi = \boxed{\phantom{\int \frac{GMm}{r^2}\,\mathrm{d}r}}$$

Exam Question Two spherical objects, of radii $R_{1,2}$ and masses $m_{1,2}$, are attracted to each other through gravity. The initial velocities of the objects are zero, the initial distance separating them is infinitely large. Find their velocities when they collide.

1.2 Three Dimension

We can use exactly the same arguments for a particle in three dimensions.

$$\mathbf{F} = -\boldsymbol{\nabla}\phi = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$$

As before multiply be $\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ and write as a total time derivative[1]

$$-\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \cdot \boldsymbol{\nabla}\phi = m\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \cdot \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$$

Integration

$$-\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m}{2}\mathbf{v}^2\right)$$

Rearrange and integrate

$$E = \frac{m}{2}\mathbf{v}^2 + \phi$$

Exam Question Consider the two-dimensional system of two particles of masses m_1 and m_2, with coordinates (x_1, y_1) and (x_2, y_2) interacting through gravity. Write down the expression for the kinetic and potential energies of the system.

Velocity of particle 1: $v_1 = \left(\boxed{}\right)$.

Velocity of particle 1: $v_2 = \left(\boxed{}\right)$.

Distance between particles: $d = \boxed{}$

Kinetic energy: $\mathrm{KE} = \boxed{} + \boxed{}$

Potential energy: $\mathrm{PE} = \boxed{}$

[1] Since ϕ is a function of x, y and z: $\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial\phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial\phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial\phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$. We can write this more compactly using the notation of vector calculus as $\frac{\mathrm{d}\phi}{\mathrm{d}t} = \mathbf{v} \cdot \boldsymbol{\nabla}\phi$.

2 Work

According to vector calculus

$$\phi_a - \phi_b = \int_b^a \boldsymbol{\nabla}\phi \cdot \mathrm{d}\mathbf{r} = \int_b^a -\mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

A useful way to calculate the potential energy of the system is to integrate from a point at which the potential energy is zero (or known).

Work is done when energy is converted from one form to another, e.g. when potential energy is converted to kinetic energy or vice versa.

Consider the change in the kinetic energy

$$\Delta \mathrm{KE} = \int_{t_1}^{t_2} \frac{\mathrm{dKE}}{\mathrm{d}t}\,\mathrm{d}t$$

By conservation of energy

$$\Delta \mathrm{KE} = \int_{t_1}^{t_2} m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\,\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$$

Replace ma by F

$$\Delta \mathrm{KE} = \int_{t_1}^{t_2} F\,\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$$

And convert to an integral over x

$$\Delta \mathrm{KE} = \int_{x_1}^{x_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

If the integral on the RHS is positive then potential energy is being converted into kinetic energy. If the integral is negative then kinetic energy is being converted to potential energy.

2.1 Frictional Forces

In a system with frictional forces work done against frictional forces gives the loss of energy from the system.

$$\Delta E = \int_{x_1}^{x_2} \mathbf{F}_{\text{fr}} \cdot \mathrm{d}\mathbf{r}$$

ΔE will be negative because frictional forces $\mathbf{F}_{\text{fr}}$ always act in the [ ] direction to motion $\mathrm{d}\mathbf{r}$.

3 Elastic Collisions

Previously we considered inelastic collisions in which bodies remain amalgamated. It is easy to see that in this case energy is not conserved. Before the collision

$$E_i = \frac{m_1}{2}v_1^2 + \frac{m_2}{2}v_2^2$$

After the collision

$$v = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$E_f = \frac{m_1 + m_2}{2}\left(\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}\right)^2$$

$$E_f = \frac{m_1^2}{2\left(m_1 + m_2\right)}v_1^2 + \frac{m_1 m_2}{m_1 + m_2}v_1 v_2 + \frac{m_2^2}{2\left(m_1 + m_2\right)}v_2^2$$

Change in kinetic energy

$$E_f - E_i = \boxed{}$$

obviously negative.

In an elastic collision, both momentum and energy are conserved. In general:

Conservation of momentum

$$m_1 \mathbf{v}_{1i} + m_2 \mathbf{v}_{2i} = m_1 \mathbf{v}_{1f} + \boxed{}.$$

Conservation of energy

$$\frac{m_1}{2}\mathbf{v}_{1i}^2 + \frac{m_2}{2}\mathbf{v}_{2i}^2 = \boxed{\qquad\qquad\qquad} + \frac{m_2}{2}\mathbf{v}_{2f}^2.$$

In principle these equations can be solved for $\mathbf{v}_{1f}$ and $\mathbf{v}_{2f}$.

Example Consider two particles with equal masses, m. Initially particle 1 moves with velocity v_0 and particle 2 is at rest. They collide elastically: what are their final velocities v_1 and v_2?

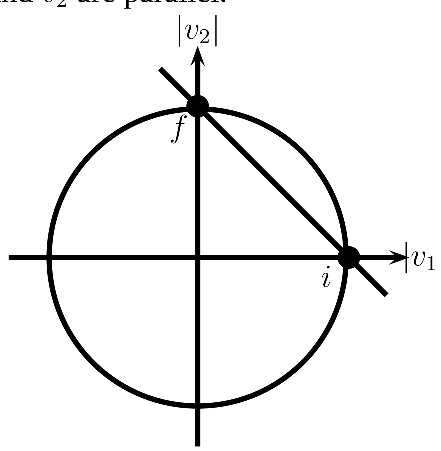
Conservation of momentum:

$$mv_0 = \boxed{\qquad\qquad} + \boxed{\qquad\qquad} \implies v_0 = \boxed{\qquad} + \boxed{\qquad}.$$

Conservation of energy:

$$\frac{m}{2}v_0^2 = \boxed{\qquad\qquad} + \boxed{\qquad\qquad} \implies v_0^2 = \boxed{\qquad} + \boxed{\qquad}.$$

Easiest to solve these equations graphically. The conservation of momentum equation describes a $\boxed{\qquad\quad}$ and the conservation of energy equation describes a $\boxed{\qquad\qquad}$. Remember, despite the way the graph is drawn, the system is one dimensional and v_1 and v_2 are parallel.



In the final state, all of the velocity of the moving particle has been transferred to the stationary particle: $v_1 = 0$, $v_2 = v_0$.

This is something you sometimes see in snooker or pool where one ball strikes another head on.

Example How would my graphical solution look if the collision was inelastic and the particles coalesced?

3.1 Newton's Cradle

In Newton's cradle collisions are elastic. The motion of the balls in the cradle can be understood be breaking the motion down as individual collisions, in which momentum is exchanged between the balls until it reaches the ball at the end of the row.