Conservation of Energy

MS4414 Theoretical Mechanics

William Lee

Contents

1 Introduction

Conserved quantities are useful in mechanics: they allow us to make short-cuts in calculations.

For example we have already encountered conservation of momentum. The principle of conservation of momentum states that the momentum of a collection of particles is a conserved quantity, provided the particles do not interact with any force fields. In collisions the momentum of particles before and after the collisions are the . This allows us calculate the velocities of the particles after the collision without knowing any of the details of the forces acting between them (which are unbelievably complicated and still the subject of research).

Consevation of energy is similarly useful. Here we will look at conservation of energy in a uniform gravitational field. We will see that, if we are interested only in positions and velocities, conservation of energy offers a simpler way to solve problems.

We will revisit conservation of energy when we discuss elastic collisions, nonuniform gravitational fields, and the Lagrangian and Hamiltonian mechanics.

2 Uniform Gravitational field

Consider a mass falling in a gravitational field $\mathbf{g} = (0, -g)$. In this case momentum [ ] conserved. We will use Newton's laws of motion to show that there is a function of the position and velocity of the particle which is conserved i.e. its derivative with respect to time is [ ]. We will call the constant of integration the *energy*.

Newton's second law states that

$$m\mathbf{g} = m\mathbf{a} \tag{1}$$

Take the dot product of both sides with the vector $\mathbf{v}$, the velocity of the particle.

$$m\mathbf{v}\cdot\mathbf{g} = \boxed{} \tag{2}$$

Write $\mathbf{v}$ and $\mathbf{a}$ in terms of the displacement $\mathbf{s}$ of the particle.

$$m\mathbf{g}\cdot\left(\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t}\right) = m\boxed{}\cdot\boxed{} \tag{3}$$

We can rewrite both sides as total derivatives and integrate (m and $\mathbf{g}$ are constants)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\mathbf{g}\cdot\mathbf{s}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m\left(\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t}\right)^2\right] \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\mathbf{g}\cdot\mathbf{s}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}m\mathbf{v}^2\right) \tag{5}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}m\mathbf{v}^2 - m\mathbf{g}\cdot\mathbf{s}\right) \tag{6}$$

where $\mathbf{v}^2 = \boxed{}$.

As promised, we have a quantity whose derivative is zero. This means the quantity is [ ]

Integrate to obtain a constant of integration I shall call the energy, E.

$$E = \tfrac{1}{2}m\mathbf{v}^2 - m\mathbf{g}\cdot\mathbf{s} \tag{7}$$

Writing this out in component form with $\mathbf{g} = (0, -g)$, $\mathbf{v} = (v_x, v_y)$, $\mathbf{s} = (x, y)$

$$E = \frac{1}{2}m\left(v_x^2 + v_y^2\right) + mgy \tag{8}$$

$\frac{1}{2}m\mathbf{v}^2$ is called **kinetic energy**, and mgy is called **potential energy**.

Worked Example *On Earth, humans can run at up to $10\,m\,s^{-1}$ and pole vault up to about $5\,m$. Show that these involve similar energies.*

The kinetic energy of a sprinter of mass m running with velocity v is [ ]. Taking a typical mass of a sprinter as [ ], the kinetic energy generated by the sprinter is [ ].

The potential energy of a pole-vaulter at the top of the jump is [ ]. Again, taking the mass of the athlete as [ ] and acceleration due to gravity as [ ], the potential energy of the athlete is [ ]

Therefore the kinetic energy of the sprinter and the potential energy of the pole-vaulter are [ ].

Worked Example *A stone is thrown up in the air with initial velocity $v = 10\,m\,s^{-1}$. What is the maximum height reached by the stone? How high is the stone when its velocity is $-5\,m\,s^{-1}$? Take $g = 10\,m\,s^{-1}$ and neglect air resistance.*

First we will solve this problem the old fashioned way using the kinematic equations. Then we will check that using conservation of energy gives the same answers.

Kinematics The equations for the position y and velocity v of an object thrown directly upwards in a [ ] gravitational field g are

$$v = \square$$

$$y = \text{______}$$

In our case $v_0 =$ ______ and $y_0 =$ ______.

The maximum height is reached when $\frac{\mathrm{d}y}{\mathrm{d}t}$, also known as ______, is ______. This occurs at time t_1, which is given by ______. When $t = t_1$, y has obtained its ______ value, y_1. This is

$$y_1 = \text{______} = \text{______}.$$

The time at which the velocity has reached $v_2 = -5\,\mathrm{m\,s^{-1}}$ is

$$t_2 = \text{______} = \text{______}$$

At this time the height of the particle, y_2 is given by

$$y_2 = \text{______} = \text{______}.$$

Energy Now use conservation of energy to obtain the same results. The energy of the system is the sum of ______ and ______ energy.

$$E = \text{______} + \text{______}$$

Initially the height of the particle is $y_0 =$ ______ and its velocity is $v_0 =$ ______. The total energy of the particle is

$$E = \text{______}$$

When the particle has reached its maximum height, y_1 the velocity of the particle is $v_1 = \square$.

Therefore the kinetic energy of the particle is $\square$. The potential energy must be equal to the $\square$. Therefore

$$E = \frac{1}{2}mv_0^2 = \square$$

Solving for y_1 gives

$$y_1 = \square = \square \text{ m}$$

At height y_2 the velocity of the particle is $v_2 = -5\,\text{m}\,\text{s}^{-1}$. At that point the kinetic energy of the particle is

$$\square$$

and the potential energy is

$$\square$$

The sum of the potential and kinetic energies is the $\square$, therefore

$$\square = \square + \square$$

Solving this equation for y_2 gives

$$y_2 = \square = \square \text{ m}$$

Worked Example *A stone is dropped from from a height of* $125\,m$. *What is its velocity when it hits the ground? Take* $g = 10\,m\,s^{-1}$ *and neglect air resistance.*

Again, first solve this using the kinematic equations and then use energy arguments.

Kinematics We use the kinematic equation

[ ]

with $y_0 =$ [ ] m and $v_0 =$ [ ] $\mathrm{m\,s^{-1}}$ to find t_1 the time at which the stone hits the ground. At this time $y = y_1 =$ [ ] m

$$t_1 = \boxed{} = \boxed{}\ \mathrm{s}$$

Now we know the time at which the stone hits the ground we can use the kinematic equation

[ ]

with $v_0 =$ [ ] $\mathrm{m\,s^{-1}}$, to find the velocity v_1 of the stone as it hits the ground at time t_1.

$$v_2 = \boxed{} = \boxed{}\ \mathrm{m\,s^{-1}}$$

Energy Now do the same thing using energy arguments Initially the height of the stone is $y_0 =$ [ ] m and so its potential energy is [ ]. The velocity of the stone is $v_0 =$ [ ] m s and so the kinetic energy is [ ]. The total energy of the system is therefore

$$E = \boxed{}.$$

When the stone hits the ground its height is $y_1 =$ [ ] m and its velocity is v_1. The potential energy of the system is [ ] and the kinetic energy of the system is [ ]. Equating the total energy at the two points

$$\boxed{} = \boxed{}$$

Solving for v_1 gives (don't forget sign)

$$v_1 = \boxed{} = \boxed{}\ \mathrm{m\,s^{-1}}$$

Kinematics or Conservation of Energy? Where possible use conservation of energy: it is simpler. In the examples above we only used one energy equation, but needed two kinematic equations. However, it is not possible to use energy arguments to work out times—the whole point if the conservation of energy is time independence. So if the question requires you to work out, or use, a time you will have to use the kinematic equations.

In an exam (or if large sums of money or someones life depend on you getting the correct answer) calculate both ways as a check on the answer—if you have time.