Conservation of Angular Momentum

MS4414 Theoretical Mechanics

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Consider a set of particles indexed by $i = 1 \ldots n$ with masses m_i and positions $\mathbf{r}_i(t)$. The angular momentum of the system is given by

$$\mathbf{L} = \sum_i m_i \left(\mathbf{r}_i \times \mathbf{v}_i\right) \tag{1}$$

The rate of change of the angular momentum of the system

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \sum_i m_i \left(\mathbf{v}_i \times \mathbf{v}_i + \mathbf{r}_i \times \mathbf{a}_i\right) \tag{2}$$

the cross product of a vector $\mathbf{v}_i \times \mathbf{v}_i = \mathbf{0}$

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \sum_i m_i \mathbf{r}_i \times \mathbf{a}_i \tag{3}$$

Newton's second law can be used to calculate the acceleration of particle i

$$m_i \mathbf{a}_i = \sum_j \mathbf{F}_{ij} \tag{4}$$

(where $\mathbf{F}_{ii} = \mathbf{0}$). Substituting the rate of change of angular momentum

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \sum_{ij} \mathbf{r}_i \times \mathbf{F}_{ij} \tag{5}$$

I can rewrite this as

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{1}{2} \sum_{ij} \left(\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji}\right) \tag{6}$$

Newton's second law states that if body i exerts a force on body j then body j exerts an equal and oposite force on body i. Mathematically: $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$

Substituting this into the above equation we have

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{1}{2}\sum_{ij}\left(\mathbf{r}_i \times \mathbf{F}_{ij} - \mathbf{r}_j \times \mathbf{F}_{ij}\right) \tag{7}$$

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{1}{2}\sum_{ij}\left(\mathbf{r}_i - \mathbf{r}_j\right) \times \mathbf{F}_{ij} \tag{8}$$

If the force between body i and body j is acts along the line between them i.e. $\mathbf{F}_{ij} \parallel (\mathbf{r}_i - \mathbf{r}_j)$ then the cross product is zero.

$$\mathbf{F}_{ij} \parallel (\mathbf{r}_i - \mathbf{r}_j) \implies \dot{\mathbf{L}} = \mathbf{0} \tag{9}$$

$$\implies \mathbf{L} = \text{const.} \tag{10}$$

Conclusion In the case in which forces between interacting bodies act are directed along the line between those two bodies (true for gravitation, electromagnetism, and overlap forces) the angular momentum of t he system

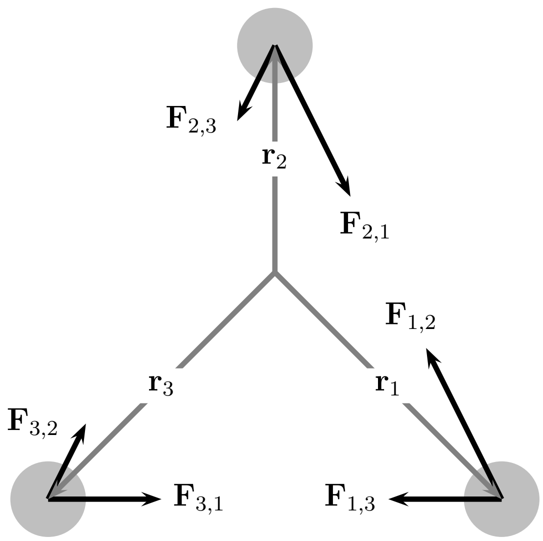
$$\mathbf{L} = \sum_i m_i\left(\mathbf{r}_i \times \mathbf{v}_i\right) \tag{11}$$

is a conserved quantity.

Examples:

- As a spinning ice skater pulls in her arms she spins faster.
- During ice ages, in which much of the worlds water is concentrated in the ice caps at the poles the days are slighly shorter.
- The moon (through the forces that create the tides) is gradually slowing down the rotation of the Earth. The moon is also moving further away from the earth.

Exam Question 2007r Consider a system of three particles of masses m_1, m_2 and m_3 with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, interacting with forces $\mathbf{F}_{1,2}$, $\mathbf{F}_{2,1}$, $\ldots$, $\mathbf{F}_{3,2}$. Prove that the angular momentum of the system with respect to the origin is conserved.



Angular momentum of the system

$$\mathbf{L} = m_1\left(\mathbf{r}_1 \times \mathbf{v}_1\right) + m_2\left(\mathbf{r}_2 \times \mathbf{v}_2\right) + m_3\left(\mathbf{r}_3 \times \mathbf{v}_3\right)$$

Rate of change of angular momenum

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = m_1\left(\mathbf{r}_1 \times \mathbf{a}_1\right) + m_2\left(\mathbf{r}_2 \times \mathbf{a}_2\right) + m_3\left(\mathbf{r}_3 \times \mathbf{a}_3\right)$$

Newton's second law

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r}_1 \times \left(\mathbf{F}_{1,2} + \mathbf{F}_{1,3}\right) + \mathbf{r}_2 \times \left(\mathbf{F}_{2,3} + \mathbf{F}_{2,1}\right) + \mathbf{r}_3 \times \left(\mathbf{F}_{3,1} + \mathbf{F}_{3,2}\right)$$

Newton's third law

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r}_1 \times \left(\mathbf{F}_{1,2} - \mathbf{F}_{3,1}\right) + \mathbf{r}_2 \times \left(\mathbf{F}_{2,3} - \mathbf{F}_{1,2}\right) + \mathbf{r}_3 \times \left(\mathbf{F}_{3,1} - \mathbf{F}_{2,3}\right)$$

Rearrange terms

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \left(\mathbf{r}_1 - \mathbf{r}_2\right) \times \mathbf{F}_{1,2} + \left(\mathbf{r}_2 - \mathbf{r}_3\right) \times \mathbf{F}_{2,3} + \left(\mathbf{r}_3 - \mathbf{r}_1\right) \times \mathbf{F}_{3,1}$$

Each term is a cross product between two parallel vectors and is thus zero.

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{0}$$

I.e. conservation of angular momentum.