Laplace Transforms

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1 Definition

1.1 Introduction

The Laplace transform is a functional operator, i.e. an operator which associates to a function a new function called the Laplace transformed function.

An important feature of the Laplace transform is the fact that it changes derivatives into multiplications by the variable. This can be very useful for solving differential equations since the Laplace transform turns them into algebraic equations.

1.2 Definition

For a given function f depending on the variable t, we define its Laplace transform F depending on the variable s, denoted also by $\mathcal{L}[f]$, as follows:

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}dt \tag{1}$$

The Laplace transform of a given function can be defined only when the integral in (1) is finite. It is the case for a large class of functions when $s > 0$.

1.3 Examples

(i) $f(t) = 1$. The Laplace transform is defined for $s > 0$ and is given by:

$$\begin{aligned} F(s) = \mathcal{L}[f](s) =& \mathcal{L}[1] = \int_0^\infty e^{-st} dt \\ =& \left[\frac{e^{-st}}{-s} \right]_0^\infty \\ =& -\frac{1}{s}(0-1) \\ =& \frac{1}{s} \end{aligned}$$

(ii) $f(t) = t$. The Laplace transform is defined for $s > 0$ and is given by:

$$\begin{aligned} F(s) =& \int_0^\infty t e^{-st} dt \\ =& \left[t\frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} dt \\ =& -\frac{1}{s} \left[te^{-st} \right]_0^\infty - \frac{1}{s^2} \left[e^{-st} \right]_0^\infty \end{aligned}$$

For $s > 0$,

$$\begin{aligned} \lim_{t \to \infty} te^{-st} =& \lim_{t \to \infty} \frac{t}{e^{st}} \\ =& 0 \end{aligned}$$

We get then

$$\begin{aligned} F(s) =& (0-0) - \left(0 - \frac{1}{s^2} \right) \\ =& \frac{1}{s^2} \end{aligned}$$

(iii) Starting from the previous example and using induction, one can prove that for any $n \in \mathbb{N}$,

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \qquad \text{for } s > 0. \tag{2}$$

(iv) $f(t) = e^{at}$. The Laplace transform is defined for $s > a$ and is given by:

$$\begin{aligned}
F(s) = \mathcal{L}[f](s) =& \int_0^\infty e^{at}e^{-st}dt \\
=& \int_0^\infty e^{(a-s)t}dt \\
=& \left[\frac{e^{(a-s)t}}{a-s}dt\right]_0^\infty \\
=& \frac{1}{s-a}
\end{aligned}$$

(v) The heaviside or unit step function. For $s > 0$:

$$f(t) = \mathcal{U}_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

$$\begin{aligned}
\mathcal{L}[\mathcal{U}_a](s) =& \int_0^\infty \mathcal{U}_a(t)e^{-st}dt \\
=& \int_0^a 0.e^{-st}dt + \int_a^\infty 1.e^{-st}dt \\
=& \left[\frac{e^{-st}}{-s}\right]_a^\infty \\
=& 0 - \frac{e^{-as}}{-s} \\
=& \frac{e^{-as}}{s}
\end{aligned}$$

1.4 The inverse Laplace transform

The inverse Laplace transform of a given function F depending, is the function f satisfying:

$$\mathcal{L}[f] = F \tag{3}$$

We denote the inverse Laplace transform by $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}[F] = f \tag{4}$$

The inverse Laplace transforms of classic functions can be found reading the tables backwards.

1.4.1 Example

Since
$$\mathcal{L}[e^{2t}](s) = \frac{1}{s-2},$$
we have:
$$\mathcal{L}^{-1}[\frac{1}{s-2}](s) = e^{2t}.$$

2 Properties

2.1 Linearity

For any functions f and g and any real λ and μ, we have:
$$\mathcal{L}[\lambda f + \mu g] = \lambda \mathcal{L}[f] + \mu \mathcal{L}[g] \tag{5}$$

2.1.1 Application

This property can be useful for finding $\mathcal{L}[\cosh]$ as follows:
$$\begin{aligned}
\mathcal{L}[\cosh](s) =& \mathcal{L}\left[\frac{e^t + e^{-t}}{2}\right] \\
=& \frac{1}{2}\mathcal{L}[e^t] + \frac{1}{2}\mathcal{L}[e^{-t}] \\
=& \frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s+1}\right) \\
=& \frac{s}{s^2-1}
\end{aligned}$$

We can also use this property to find $\mathcal{L}[\cos(\omega t)]$ and $\mathcal{L}[\sin(\omega t)]$ using complex numbers. In fact, since
$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t),$$
recalling that:
$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a},$$

for $s > \Re(a)$, letting $a = i\omega$, we have for any

$$\begin{aligned}
\mathcal{L}[\cos(\omega t)] + i\mathcal{L}[\sin(\omega t)] =& \mathcal{L}[\cos(\omega t) + i\sin(\omega t)] \\
=& \mathcal{L}[e^{i\omega t}] \\
=& \frac{1}{s - i\omega} \\
=& \frac{s + i\omega}{(s - i\omega)(s + i\omega)} \\
=& \frac{s}{s^2 + \omega^2} + i\frac{\omega}{s^2 + \omega^2}
\end{aligned}$$

Then, by identifying the real and imaginary parts, we get:

$$\begin{aligned}
\mathcal{L}[\cos(\omega t)] &= \frac{s}{s^2 + \omega^2} \\
\mathcal{L}[\sin(\omega t)] &= \frac{\omega}{s^2 + \omega^2}
\end{aligned}$$

The linearity of the Laplace transform implies the linearity of the inverse Laplace transform. Therefore, the inverse Laplace transform of $F(s) = \frac{s+2}{s^2+1}$ is given by:

$$\begin{aligned}
\mathcal{L}^{-1}[F](t) =& \mathcal{L}^{-1}\left[\frac{s}{s^2+1} + \frac{2}{s^2+1}\right] \\
=& \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + \mathcal{L}^{-1}\left[\frac{2}{s^2+1}\right]
\end{aligned}$$

$$= \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \cos(t) + 2\sin(t)$$

2.2 First shifting theorem

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left[e^{at}f(t)\right] = F(s - a) \tag{6}$$

This identity relates inverse Laplace transforms of a-shifted functions and is known as the *first shifting theorem*. It can be proved very easily using the definition of the Laplace transform.

2.2.1 Application

Since

$$\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}},$$

we can deduce from the first shifting theorem that:

$$\begin{aligned}\mathcal{L}\left[e^{at}t^n\right](s) =& \mathcal{L}\left[t^n\right](s-a)\\ =& \frac{n!}{(s-a)^{n+1}}\end{aligned}$$

2.3 Second shifting theorem

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}^{-1}\left[e^{-as}F(s)\right] = \mathcal{U}_a(t)f(t-a). \tag{7}$$

This identity is known as the *second shifting theorem.*

To prove this identity, we first look for the Laplace transform of $\mathcal{U}_a(t)f(t-a)$:

$$\begin{aligned}\mathcal{L}\left[\mathcal{U}_a(t)f(t-a)\right] =& \int_0^\infty e^{-st}\mathcal{U}_a(t)f(t-a)dt\\ =& \int_a^\infty e^{-st}f(t-a)dt\\ =& \int_0^\infty e^{-s(r+a)}f(r)dr\\ =& e^{-as}F(s),\end{aligned}$$

where $r = t-a$. Then, applying the inverse Laplace transform at both sides, we find the identity 7.

2.3.1 Application

To determine the inverse Laplace transform of $\frac{e^{-4s}}{s^3}$, we fisrt recall that

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s^3}\right] =& \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^3}\right]\\ =& \frac{1}{2}t^2 = f(t).\end{aligned}$$

Then, by the second shifting theorem, we have:

$$\begin{aligned}\mathcal{L}^{-1}\left[e^{-4s}\frac{1}{s^3}\right] =& \mathcal{U}_4(t)f(t-4)\\ =& \mathcal{U}_4(t)\frac{1}{2}(t-4)^2\\ =& \left\{\begin{array}{cc} 0 & \text{if } t<4\\ \frac{1}{2}(t-4)^2 & \text{if } t\geq 4\end{array}\right.\end{aligned}$$

2.4 Transform of a derivative

2.4.1 The first derivative

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}[f'](s) = sF(s) - f(0) \tag{8}$$

This identity can be proved using integration by parts:

$$\begin{aligned}
\mathcal{L}[f'](s) =& \int_0^\infty e^{-st} f'(t)dt \\
=& \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s)e^{-st} f(t)dt \\
=& (0 - e^{-s.0} f(0)) + s\int_0^\infty e^{-st} f(t)dt \\
=& sF(s) - f(0)
\end{aligned}$$

Note that this formula can be useful to determine $\mathcal{L}[f]$ if we know $\mathcal{L}[f']$.

2.4.2 The second derivative

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}[f''](s) = s^2 F(s) - sf(0) - f'(0) \tag{9}$$

This identity can be proved using the previous identity and an integration by parts.

2.4.3 The third derivative

In the same way, the Laplace transform of the third derivative of a function f is related to F, the Laplace transform of f, by the following identity:

$$\mathcal{L}[f^{(3)}](s) = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0) \tag{10}$$

2.4.4 The $n^{\textbf{th}}$ derivative

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0) \tag{11}$$

These formulae are useful for solving differential equations.

2.5 Transform of an integral

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left[\int_0^t f(x)dx\right] = \frac{1}{s}F(s) \tag{12}$$

This identity may be rewritten after applying the inverse Laplace transform to both sides as follows:

$$\mathcal{L}^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t f(x)dx \tag{13}$$

which is useful in finding inverse transforms.

2.6 The convolution theorem

Definition 1. *Let f and g two given functions. Then, we define the function denoted by $f * g$ and called the convolution of f and g by:*

$$(f * g)(t) = \int_0^t f(x)g(t-x)dx \tag{14}$$

Remark 1. *For any two functions f and g, we have:*

$$f * g = g * f.$$

This can be proved easily by the change of variable $u = t - x$.

Theorem 1. *Let f and g be two given functions. Then the following identity holds:*

$$\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s) \tag{15}$$

2.6.1 Application

Let H be given by:

$$H(s) = \frac{1}{(s-1)(s-2)^2}.$$

The previous theorem can be used in order to determine $\mathcal{L}^{-1}[H]$.

Let $F(s) = \frac{1}{(s-2)^2}$ and $G(s) = \frac{1}{s-1}$. Then,

$$\mathcal{L}^{-1}[F](t) = te^{2t},$$

and

$$\mathcal{L}^{-1}[G](t) = e^t.$$

By the previous theorme,

$$\begin{aligned}
\mathcal{L}^{-1}[H](t) =& \mathcal{L}^{-1}[FG](t) \\
=& \left(\mathcal{L}^{-1}[F] * \mathcal{L}^{-1}[G]\right)(t) \\
=& \int_0^t \mathcal{L}^{-1}[F](x)\mathcal{L}^{-1}[G](t-x)dx \\
=& \int_0^t xe^{2x}e^{t-x}dx \\
=& e^t \int_0^t xe^x dx \\
=& e^t \left([xe^x]_0^t - \int_0^t e^x dx \right) \\
=& e^t \left((te^t - 0) - (e^t - 1) \right) \\
=& te^{2t} - e^{2t} + e^t
\end{aligned}$$

2.7 Transform of periodic functions

Let f be a periodic function with period p. Then it can be shown that:

$$\mathcal{L}[f](s) = \frac{1}{1-e^{-ps}} \int_0^p f(t)e^{-st}dt \tag{16}$$

2.8 Further results

It can be shown that:

$$\mathcal{L}[tf(t)](s) = -\mathcal{L}[f](s),$$

and more generally:

$$\mathcal{L}[t^n f(t)](s) = (-1)^n \mathcal{L}[f](s).$$

2.9 Example

Find the inverse Laplace transform of:

$$F(s) = \frac{2s+6}{(s+1)(s^2+2s+5)}.$$

First, we express $F(s)$ as a sum of fractions:

$$F(s) = \frac{A}{s+1} + \frac{Bx+C}{s^2+2s+5}.$$

This would imply that:

$$\begin{aligned}2s+6 =& A(s^2+2s+5)+(BxC)(s+1)\\ =& (A+B)s^2+(2A+B+C)s+5A+C\end{aligned}$$

and

$$\begin{cases} A+B &= 2\\ 2A+B+C &= 2\\ 5A+C=6 \end{cases}$$

from which we deduce

$$\begin{cases} A=1\\ B=-1\\ C=1 \end{cases}$$

Then,

$$\begin{aligned}F(s) =& \frac{1}{s+1}-\frac{s-1}{s^2+2s+5}\\ =& \frac{1}{s+1}-\frac{s-1}{s^2+2s+1+4}\\ =& \frac{1}{s+1}-\frac{s-1}{(s+1)^2+4}\\ =& \frac{1}{s+1}-\frac{(s+1)-2}{(s+1)^2+4}\\ =& \frac{1}{s+1}-\frac{s+1}{(s+1)^2+4}+\frac{2}{(s+1)^2+4}\end{aligned}$$

Now, recalling that

$$\mathcal{L}[e^{at}](s)=\frac{1}{s-a},$$

we deduce

$$\mathcal{L}[e^{-t}](s)=\frac{1}{s+1}.$$

And since

$$\mathcal{L}[\cos(at)](s)=\frac{s}{s^2+a^2},$$

we have

$$\mathcal{L}[\cos(2t)](s)=\frac{s}{s^2+4}.$$

Then

$$\begin{aligned}\frac{s+1}{(s+1)^2+4} =& \mathcal{L}[\cos(2t)](s+1)\\ =& \mathcal{L}[\cos(2t)](s-(-1))\\ =& \mathcal{L}[e^{(-1).t}\cos(2t)]\end{aligned}$$

The last step is a consequence of the first shifting theorem.

Analogously, we have

$$\frac{2}{(s+1)^2+4} = \mathcal{L}[e^{(-1)t}\sin(2t)].$$

Finally,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{2s+6}{(s+1)(s^2+2s+5)}\right](t) &= e^{-t} - e^{-t}\cos(2t) + e^{-t}\sin(2t)\\ &= e^{-t}(1-\cos(2t)+\sin(2t))\end{aligned}$$

3 Application to differential equations

The Laplace transform takes functions in the t-space into the the s-space. By doing so, differential equations are transformed into algebraic equations. By solving the algebraic equation, we find the Laplace transform of the solution to the differential equation. Then the solution of the differential equation can be found by moving back in the t-space using the inverse Laplace transform.

3.0.1 Example 1

Consider the following differential equation:

$$y' + 2y = e^{-2t}, \qquad y(0) = 3.$$

Then defining $Y = \mathcal{L}[y]$, we have:

$$\begin{aligned}\mathcal{L}[y' + 2y] =& \mathcal{L}[y'] + 2\mathcal{L}[y]\\ =& sY - y(0) + 2Y\\ =& (s+2)Y - y(0)\end{aligned}$$

and

$$\mathcal{L}[e^{-2t}](s) = \frac{1}{s+2}.$$

Applying the Laplace transform to the differential equation leads to the following algebraic equation:

$$(s+2)Y - 3 = \frac{1}{s} + 2,$$

and the solution is:

$$Y = \frac{3}{s+2} + \frac{1}{(s+2)^2}.$$

The solution y to the differential equation is then:

$$\begin{aligned}
y(t) =& \mathcal{L}^{-1}\left[\frac{3}{s+2}+\frac{1}{(s+2)^2}\right]\\
=& 3\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]+\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right]\\
=& 3e^{-2t}+te^{-2t}\\
=& (t+3)e^{-2t}
\end{aligned}$$

3.0.2 Example 2

Consider the following differential equation:

$$y''+4y=4t+4, \qquad y(0)=3, y'(0)=-1.$$

After applying the Laplace transform, we get

$$(s^2Y-sy(0)-y'(0))+4Y=\frac{4}{s^2}+\frac{4}{s}.$$

It follows that:

$$\begin{aligned}
Y(s) =& \frac{3s^3-s^2+4s+4}{s^2(s^2+4)}\\
=& \frac{1}{s}+\frac{1}{s^2}+2\frac{s-1}{s^2+4}\\
=& \frac{1}{s}+\frac{1}{s^2}+2\frac{s}{s^2+4}-\frac{2}{s^2+4}
\end{aligned}$$

Then

$$y(t)=1+t+2\cos(2t)-\sin(2t).$$