Ordinary Differential Equations

Sofiane Soussi
and William Lee
william.lee@ul.ie
http://www.ul.ie/wlee/ma4005.html

November 8, 2011

Differential equations are equations where the unknown is a function and which involve derivatives of the unknown function of various orders.

Khan Academy. What is a differential equation?

1 Differential equations of 1st order

1.1 Differential equations with separable variables

Khan Academy Separable differential equations.

1.1.1 Example

Let us consider the following differential equation:

$$y' = \frac{1}{x\tan(y)}.$$

Here y is a function of x : $y(x)$ and y' denotes the derivative of y with respect to x, i.e.

$$y'(x) = \frac{dy}{dx}.$$

This equations can be rewritten in the following way:

$$y'\tan(y) = \frac{1}{x},$$

Khan Academy. Separable differential equations 2.

or more explicitly:

$$\frac{y'\sin(y)}{\cos(y)} = \frac{1}{x}.$$

Now we remark that

$$\frac{d}{dx}\Big(\cos(y)\Big) = -y'\sin(y)$$

which means that:

$$\frac{y' \sin(y)}{\cos(y)} = -\frac{d}{dx}\bigg(\ln(\cos(y))\bigg).$$

Finally we can integrate both sides of the differential equation as follows:

$$\int \frac{y' \sin(y)}{\cos(y)} dx = \int \frac{1}{x} dx,$$

and we get:

$$-\ln(\cos(y)) = \ln(x) + c,$$

where c is some constant. Now taking the exponential of both sides, we get:

$$\frac{1}{\cos(y)} = e^c x,$$

which leads to:

$$y = \cos^{-1}\left(\frac{\lambda}{x}\right),$$

here λ is a constant (replacing e^{-c}).

1.1.2 Example

The following is another example which is solved by the same method but using short-hand notations:

$$\frac{dy}{dx} = 2xy.$$

Taking y to the left-hand side and the x to the right-hand side, we get:

$$\frac{dy}{y} = 2x dx,$$

and after integration, we get:

$$\ln(y) = x^2 + c.$$

Taking the exponential of both sides and denoting e^c by λ, we get

$$y = \lambda e^{x^2}.$$

The *separation of variables* is a method that can be applied whenever it is possible to reformulate the differential equation taking the unknown function (and its derivatives) into one side of the equation and the variable (x) into the other side.

In the following equation, the separation of variables cannot be applied.

$$y' - \sin(xy) = 0.$$

Sometimes the separation of variables is not very obvious as in the following example.

$$e^{xy'} = x,$$

where taking the $\ln$, we get:

$$xy' = \ln(x).$$

1.2 Linear differential equations

1.2.1 Introduction

Linear differential equations are differential equations where the unknown functions and its derivatives are only multiplied by some coefficients and summed together.

The order of the differential equations is the order of the highest derivative that appears in it.

The following differential equation is a linear differential equation of order 3.

$$y^{(3)} - \sin(x)y'' + 5y' + \ln(x)y = e^x,$$

The coefficients of this linear differential equation are 1 (for $y^{(3)}$), $-\sin(x)$ (for y''), 5 (for y'), and $\ln(x)$ (for y).

The right-hand side is equal to e^x.

The following differential equation is a non linear differential equation of order 2.

$$(y'')^2 + \cos(x)y' - \sin(y) = e^x.$$

It is non linear because the second derivative appears with a power 2 and the 0 order derivative appears inside the $\sin$ function.

1.2.2 Homogeneous linear differential equations with constant coefficients

A homogeneous linear differential equation is a differential equation where the right-hand side is equal to 0. The two previous examples are non homogeneous differential equations as the right-hand side was e^x in both.

A typical homogeneous first-order linear differential equation with constant coefficients is:

$$y' + ay = 0, \tag{1}$$

where $a \in \mathbb{R}$ is a constant. To solve this differential equation, we separate the variables as follows:

$$\frac{dy}{dx} = -ay,$$
$$\frac{dy}{y} = -adx,$$

then we integrate both sides:

$$\int \frac{dy}{y} = -a \int dx + c, \qquad c \in \mathbb{R},$$

which leads to:

$$\ln|y| = -ax + c$$

Then we take the exponential of both sides and we get:

$$|y| = e^c e^{-ax}$$

and then

$$y = \lambda e^{-ax}, \qquad \lambda \in \mathbb{R}.$$

This is the general solution to equation (1).

1.2.3 Homogeneous linear differential equations with non constant coefficients

Wikipedia
Linear differential equations: First order equation.

A typical homogeneous first-order linear differential equation with non constant coefficients is:

$$y' + a(x)y = 0, \tag{2}$$

The difference between this differential equation and (1) is that the coefficient is not a constant a but a function $a(x)$.

An example is the following:

$$y' + \cos(x)y = 0 \tag{3}$$

To solve equation (2) we can separate the variables as done for (1) and end with:

$$\int \frac{dy}{y} = -\int a(x)dx + c, \qquad c \in \mathbb{R},$$

which leads to:

$$\ln|y| = -\int a(x)dx + c$$

Then taking the exponential, we get:

$$y = \lambda e^{-\int a(x)dx}, \qquad \lambda \in \mathbb{R}.$$

This is the general solution to equation (2).

The general solution to equation (3) is then:

$$\begin{aligned} y &= \lambda e^{-\int \cos(x)dx} \\ &= \lambda e^{-\sin(x)}, \qquad \lambda \in \mathbb{R}. \end{aligned}$$

1.2.4 Non homogeneous linear differential equations

A typical non homogeneous first-order linear differential equation with non constant coefficients is:

$$y' + a(x)y = b(x), \tag{4}$$

The differential equation is non homogeneous because the right-hand side is $b(x)$ and not 0.

To solve this differential equation, we use the method of the "variation of the constant" where we take the general solution to the homogeneous equation:

$$y = \lambda e^{-\int a(x)dx}, \qquad \lambda \in \mathbb{R}.$$

Wikipedia Variation of parameters.

and replace the constant λ by a function $\lambda(x)$.

It follows that the derivative y' of y is given by:

$$y' = \lambda'(x)e^{-\int a(x)dx} + a(x)\lambda(x)e^{-\int a(x)dx},$$

then we check how y can be a solution to (4):

$$\begin{aligned} y' + a(x)y &= \lambda'(x)e^{-\int a(x)dx} + a(x)\lambda(x)e^{-\int a(x)dx} + a(x)\lambda e^{-\int a(x)dx} \\ &= \lambda'(x)e^{-\int a(x)dx} \end{aligned}$$

This means that y is a solution to (4) if and only if:

$$\lambda'(x)e^{-\int a(x)dx} = b(x),$$

or

$$\lambda'(x) = b(x)e^{\int a(x)dx},$$

that is equivalent to:

$$\lambda(x) = \int b(x)e^{\int a(x)dx}dx + c, \qquad c \in \mathbb{R}.$$

The general solution to (4) is then:

$$y = \left(\int b(x)e^{\int a(x)dx}dx\right)e^{-\int a(x)dx} + ce^{-\int a(x)dx}, \qquad c \in \mathbb{R}.$$

We see clearly that the solution y is the sum of two functions, the first one is

$$y_I = \left(\int b(x)e^{\int a(x)dx}dx\right)e^{-\int a(x)dx},$$

is a particular solution to (4) also called *a particular integral*, and the second one

$$y_g = ce^{-\int a(x)dx},$$

is the general solution to the homogeneous equation

$$y' + a(x)y = 0.$$

An alternative but less general method is to deduce the right answer by the structure of the equation (often to within an undetermined parameter or two). This is known as the *method of undetermined coefficients* or the *lucky guess method*.

Wikipedia Method of undetermined coefficients.

1.2.5 Example

Let us consider the following differential equation:

$$y' + 3\cos(x)y = \cos(x).$$

Wolfram Alpha Example solved. If using qrcode type: `dy/dx=3*cos(x)*y=cos(x)` QR code generator mangles input: sorry!

This is a non homogeneous linear differential equation of first order with non constant coefficients.

Following the previous paragraph, the solution is given by:

$$\begin{aligned}
y =& \left(\int \cos(x)e^{3\int\cos(x)dx}dx\right)e^{-3\int\cos(x)dx} + ce^{-3\int\cos(x)dx}, \qquad c \in \mathbb{R}, \\
=& \left(\int \cos(x)e^{3\sin(x)}dx\right)e^{-3\sin(x)} + ce^{-3\sin(x)} \\
=& \frac{1}{3}\left(\int de^{3\sin(x)}\right)e^{-3\sin(x)} + ce^{-3\sin(x)} \\
=& \frac{1}{3}e^{3\sin(x)}e^{-3\sin(x)} + ce^{-3\sin(x)} \\
=& \frac{1}{3} + ce^{-3\sin(x)}
\end{aligned}$$

Alternatively one can remark that the constant function $y = \frac{1}{3}$ is a particular solution to the differential equation, determine the general solution go the homogeneous equation

$$y' + 3\cos(x)y = 0$$

and finally write the general solution to the non homogeneous equation as the sum of the particular solution and the general solution to the homogeneous equation.

Exam Question Find the general solutions of the differential equations (i) $\frac{dy}{dx} + 2xy = e^{1-x^2}$; (ii) $y'' - y' = \sin x + \cos x$. *2003/4.*

Exam Question Find the general solution of the differential equation

$$\frac{dy}{dx} + xy = xy^2$$

The current in a circuit is given by

$$L\frac{dI}{dt} + RI = E$$

where L, R and E are constants. Find the general solution for I in terms of t and the particular solution for which $I = I_0$ at $t = 0$. Show that the terminal value of the current is E/R. *2004/5.*

2 The Runge-Kutta method

2.1 Introduction

In many problems, we face complicated differential equations that we cannot solve analytically. But sometimes we are just interested in an approximation of a particular solution on a fixed interval of finite length.

It is possible then to have an approximation of the solution using appropriate numerical methods.

One important method is the Runge-Kutta method. It is used to find approximations to solutions of general first-order differential equations of the form:

$$y' = f(x, y), \tag{5}$$

where f is some given function of 2 variables.

2.1.1 Example

The following is an example of differential equations one can solve numerically with the Runge-Kutta method:

$$y' = \cos(x^2 y)e^{x^2+y^2}.$$

In this example the function f is given by:

$$f(x, y) = \cos(x^2 y)e^{x^2+y^2}.$$

2.2 The Runge-Kutta method

Suppose we are looking for the particular solution y of equation (5) satisfying $y(a) = y_0$ where y_0 is a given real number. And suppose that we are interested in a numerical approximation of the solution over an interval $[a, b]$.

Then, we first divide the interval $[a, b]$ is N subdivisions of equal length $h = \frac{b-a}{N}$ where N is a given integer. The subdivisions will be the intervals $[x_n, x_{n+1}]$, $0 \leq n \leq N$ where

$$x_n = a + hn = a + \frac{b-a}{N}n.$$

This means that the first point $x_0 = a$ and the last point $x_{N+1} = b$.

Then we look for $(y_n)_{0 \leq n \leq N+1}$ which are the numerical approximations of $(y(x_n))_{0 \leq n \leq N+1}$, i.e. approximations of the exact solution y at the points $(x_n)_{0 \leq n \leq N+1}$.

The first value y_0 is given in the problem.

Suppose that we have an approximation of y_n, then in order to get an approximation of y_{n+1}, we first determine k_1, k_2, k_3, and k_4 as follows:

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \tfrac{h}{2}, y_n + \tfrac{1}{2}k_1) \\ k3 &= hf(x_n + \tfrac{h}{2}, y_n + \tfrac{1}{2}k_2) \\ k_4 &= hf(x_n + h, y_n + k_3) \end{aligned}$$

Then we define Δy_n as:

$$\Delta y_n = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right),$$

and we increment y_n by Δy_n:

$$y_{n+1} = y_n + \Delta y_n.$$

Since we know y_0, we can apply this method to find an approximation of y_1. Then using the same method with x_1 and y_1, we get an approximation of y_2, and so on. At the end we get an approximation of y_{N+1}.

2.2.1 Example

Consider the following differential equation:

$$y' = x + y,$$

and let us look for an approximation of the particular solution satisfying $y(0) = 1$ on the interval $[0, 1]$ using 2 steps. ($N = 2$)

In this case $f(x, y) = x + y$, $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $y_0 = 1$, and $h = \frac{1-0}{2} = 0.5$.

In order to find an approximation of y_1, we first determine k_1, k_2, k_3, and k_4:

$$\begin{aligned} k_1 &= hf(x_0, y_0) = 0.5 \\ k_2 &= hf(x_0 + \tfrac{h}{2}, y_0 + \tfrac{1}{2}k_1) = 0.75 \\ k3 &= hf(x_0 + \tfrac{h}{2}, y_0 + \tfrac{1}{2}k_2) = 0.8125 \\ k_4 &= hf(x_0 + h, y_0 + k_3) = 0.15625 \end{aligned}$$

Then we determine Δy_0:

$$\Delta y_0 = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 0.796875,$$

and then :

$$y_1 = y_0 + \Delta y_0 = 1 + 0.796875 = 1.796875$$

Then we apply determine the new values of k_1, k_2, k_3, and k_4 using $x_1 = 0.5$ and $y_1 = 1.796875$ instead of x_0 and y_0. We find:

$$\begin{aligned} k_1 &= hf(x_1, y_1) = 1.1485 \\ k_2 &= hf(x_1 + \tfrac{h}{2}, y_1 + \tfrac{1}{2}k_1) = 1.561 \\ k3 &= hf(x_1 + \tfrac{h}{2}, y_1 + \tfrac{1}{2}k_2) = 1.664 \\ k_4 &= hf(x_1 + h, y_1 + k_3) = 2.106 \end{aligned}$$

which leads to:

$$\Delta y_1 = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 1.1617,$$

and:

$$y_2 = y_1 + \Delta y_1 = 1.796875 + 1.1617 = 3.141$$

3 Linear second-order differential equations with constant coefficients

3.1 Homogeneous linear second-order differential equations with constant coefficients

The general form a such differential equations is

$$ay'' + by' + cy = 0, \tag{6}$$

where $a \neq 0$, b, and c are constants.

Khan Academy Second order homogeneous linear equations 1.

When a, b, c are positive this equation describes a damped harmonic oscillator. This could be a mass $m = a$, on a spring $k = c$ with damping $\mu = b$, or an LCR circuit $L = a$, $R = b$, $C = c^{-1}$.

If we formally replace $y^{(n)}$(n^{th} derivative of the function y) by r^n (r to the power n), we get the so-called characteristic equation of the differential equation:

$$ar^2 + br + c = 0, \tag{7}$$

which is a polynomial equation of degree 2. The discriminant of this equation is given by:

$$\Delta = b^2 - 4ac.$$

Wolfram Mathworld Damped simple harmonic motion.

The solution is given in one the following cases depending on Δ:

Two real roots $\Delta > 0$: then equation (7) has 2 distinct real roots r_+ and r_- given by

$$r_{\pm} = \frac{-b \pm \sqrt{\Delta}}{2a},$$

Khan Academy Characteristic polynomial, $y'' + 5y' + 6 = 0$.

and the general solution to the differential equation (6) is given by:

$$y = \alpha e^{r_+ x} + \beta e^{r_- x},$$

where α and β are arbitrary constants.

Repeated roots $\Delta = 0$: then equation (7) has one double real root

$$r_0 = -\frac{b}{2a},$$

and the general solution to the differential equation (6) is given by:

$$y = e^{r_0 x}(\alpha x + \beta),$$

where α and β are arbitrary constants.

Khan Academy
Repeated root.

Complex conjugate roots $\Delta < 0$: then equation (7) has 2 conjugate complex roots r_+ and $r_- = \bar{r_+}$ given by

$$r_\pm = \frac{-b \pm i\sqrt{-\Delta}}{2a},$$

where $i = \sqrt{-1}$. If we define $p = \frac{-b}{2a}$ (the real part of r_+ and r_-) and $q = \frac{\sqrt{\Delta}}{2a}$ (the imaginary part of r_+, then we can rewrite the solutions to (7) as

$$r_\pm = p \pm iq.$$

Then the general solution to the differential equation (6) is given by:

$$y = e^{px}\left(\alpha \cos(qx) + \beta \sin(qx)\right),$$

where α and β are arbitrary constants.

Khan Academy
Complex roots.

Khan Academy
Complex roots contd.
$y'' + y' + 1 = 0$.

Khan Academy
$y'' + 5y' + 6 = 0$,
$y(0) = 2$, $y'(0) = 3$.

3.1.1 Examples

- Example 1 : $\Delta > 0$

$$y'' - 5y' + 4y = 0.$$

  The solutions to the characteristic equation

$$r^2 - 5r + 4 = 0$$

  are 1 and 5. The general solution to the differential equation is then given by:

$$y = \alpha e^x + \beta e^{5x}.$$

Khan Academy
$4y'' - 9y' + 3y = 0$,
$y(0) = 2$, $y'(0) = \frac{1}{2}$.

- Example 2 : $\Delta = 0$

$$y'' - 2y' + y = 0.$$

The characteristic equation is

$$r^2 - 2r + 1 = (r-1)^2 = 0$$

which has only one double root 1. The general solution to the differential equation is then given by:

$$y = e^x(\alpha x + \beta).$$

Khan Academy $y'' - y' + \frac{1}{4} = 0$, $y(0) = 2$, $y'(0) = \frac{1}{3}$

- Example 3 : $\Delta < 0$

$$y'' - 2y'5y = 0.$$

The characteristic equation is

$$r^2 - 2r + 5 = 0$$

The discriminant is $\Delta = (-2)^2 - 4(1)(5) = -16$. The roots of the characteristic equation are:

$$r_{\pm} = \frac{-(-2) \pm \sqrt{-16}}{2(1)} = 1 \pm 2i$$

The general solution to the differential equation is then given by:

$$y = e^x(\alpha \cos(2x) + \beta \sin(2x)).$$

Khan Academy $y'' + 4y' + 5y = 0$, $y(0) = 1$, $y'(0) = 0$.

3.2 Non-homogeneous linear second-order differential equations with constant coefficients

In the non-homogeneous case:

$$ay'' + by' + cy = f(x), \tag{8}$$

where f is a given function, the general solution if given by:

$$y = y_I + \alpha y_1 + \beta y_2, \tag{9}$$

where y_I is a particular solution, also called *particular integral* of (8), y_1 and y_2 are 2 linearly independent solutions of the homogeneous equation, α and β are constants. ($\alpha y_1 + \beta y_2$ is the general solution to the homogeneous equation)

Khan Academy Inhomogeneous equations.

Since the solution of the homogeneous equation can be found from the previous paragraph, we only need to find a particular solution to (8) in order to find all the solutions.

Here we present a method for finding a particular solution in some particular cases.

3.2.1 Case of a polynomial right-hand side

In the case where the right-hand side is a polynomial $P_n(x)$ of degree n,

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

we look for a particular integral as a polynomial of the same degree,

$$y_I = \mu_n x^n + \mu_{n-1}x^{n-1} + \ldots + \mu_1 x + \mu_0,$$

which implies that:

$$y_I' = n\mu_n x^{n-1} + (n-1)\mu_{n-1}x^{n-2} + \ldots + 2\mu_2 x + \mu_1,$$

and

$$y_I'' = n(n-1)\mu_n x^{n-2} + (n-1)(n-2)\mu_{n-1}x^{n-3} + \ldots + 6\mu_3 x + 2\mu_2.$$

Khan Academy
$y'' - 3y' - 4y = 4x^2$.

Then we substitute the expressions of y_I, y_I' and y_I'' in the differential equation and deduce the coefficients μ_0, μ_1,..., μ_n.

- Example:

  $$y'' + 4y = x^2$$

  The general solution of the homogeneous equation is given by:

  $$y = \alpha\cos(2x) + \beta\sin(2x).$$

  Then we look for a particular integral

  $$y_I = \mu_2 x^2 + \mu_1 x + \mu_0.$$

  Since

  $$y_I' = 2\mu_2 x + \mu_1,$$

  and

  $$y_I'' = 2\mu_2,$$

  we have

  $$(2\mu_2) + 4(\mu_2 x^2 + \mu_1 x + \mu_0) = x^2,$$

  and then

  $$(4\mu_2 - 1)x^2 + 4\mu_1 x + 4\mu_0 + 2\mu_2 = 0.$$

  It follows that:

  $$\begin{cases} 4\mu_2 - 1 = 0 \\ 4\mu_1 = 0 \\ 4\mu_0 + 2\mu_2 = 0 \end{cases}$$

  Finally we get:

  $$\begin{cases} \mu_2 = \frac{1}{4} \\ \mu_1 = 0 \\ \mu_0 = -\frac{1}{8} \end{cases}$$

  The general solution to the integral equation is then given by:

  $$y = \frac{x^2}{4} - \frac{1}{8} + \alpha\cos(2x) + \beta\sin(2x), \qquad \alpha, \beta \in \mathbb{R}.$$

3.2.2 Case of an exponential right-hand side

Here we study the case where the right-hand side is a product of a polynomial and an exponential, $P_n(x)e^{\gamma x}$, P_n being a polynomial of degree n having the same expression as in the previous paragraph.

In this case, we look for a particular integral having the form:

$$y_I = Q_n(x)e^{\gamma x},$$

where Q_n is a polynomial of the same degree as P_n, i.e.

$$Q_n = \mu_n x^n + \mu_{n-1}x^{n-1} + \ldots + \mu_1 x + \mu_0.$$

Then we determine y_I' and y_I'' and substitute the results in the differential equation in order to find the undetermined coefficients μ_0, μ_1,..., μ_n.

- Example:

  $$y'' - 4y = xe^{3x}$$

  The general solution of the homogeneous equation is given by:

  $$y = \alpha e^{-2x} + \beta e^{2x}.$$

  Then we look for a particular integral

  $$y_I = (\mu_1 x + \mu_0)e^{3x}.$$

  Since

  $$y_I' = (3\mu_1 x + \mu_1 + 3\mu_0)e^{3x},$$

  and

  $$y_I'' = (9\mu_1 x + 6\mu_1 + 9\mu_0)e^{3x},$$

  we have

  $$(5\mu_1 x + 6\mu_1 + 5\mu_0)e^{3X} = xe^{3x},$$

  and then It follows that:

  $$\begin{cases} 5\mu_1 = 1 \\ 6\mu_1 + 5\mu_0 = 0 \end{cases}$$

  Finally we get:

  $$\begin{cases} \mu_1 = \frac{1}{5} \\ \mu_0 = -\frac{6}{25} \end{cases}$$

  The general solution to the integral equation is then given by:

  $$y = \left(\frac{1}{5}x - \frac{6}{25}\right)e^{3x} + \alpha e^{-2x} + \beta e^{2x}, \qquad \alpha, \beta \in \mathbb{R}.$$

- In the case where the right-hand side is a solution to the homogeneous differential equation as in the following example:

$$y'' + 2y = xe^{2x},$$

this method does not work.

In fact, if we try to get a particular integral

$$y_I = (\mu_1 x + \mu_0)e^{2x},$$

and we replace in the differential equation, we get

$$4\mu_1 e^{2x} = xe^{2x},$$

which is impossible. This means that there exists no solution to the differential equation having that form.

In such case we need to look for a particular integral having the form of a product of a polynomial and an exponential functions, but where the degree of the polynomial is equal to the degree of the polynomial in the right-hand side plus 1.

This means that in our example, we need to look for a particular integral having the expression:

$$y_I = (\mu_2 x^2 + \mu_1 x + \mu_0)e^{2x},$$

which implies that:

$$y_I'' = \Big(4\mu_2 x^2 + 4(\mu_1 + 2\mu_2)x + 4(\mu_0 + \mu_1 + 2\mu_2)\Big)e^{2x},$$

then, substituting in the differential equation, we get

$$\Big(8\mu_2 x + 4(\mu_1 + 2\mu_2)\Big)e^{2x} = xe^{2x},$$

then,

$$\begin{cases} 8\mu_2 = 1 \\ 4(\mu_1 + 2\mu_2) = 0 \end{cases}$$

and

$$\begin{cases} \mu_2 = \frac{1}{8} \\ \mu_1 = -\frac{1}{4} \end{cases}$$

The general solution to the differential equation is then,

$$y = \left(\frac{x^2}{8} - \frac{x}{4} + \alpha\right)e^{2x} + \beta e^{-2x}, \qquad \alpha, \beta \in \mathbb{R}.$$

3.2.3 Case of a right-hand side with $\cos$ and $\sin$ functions.

In the case of a right-hand side having the form

$$P_n(x)\cos(\gamma x) \qquad \text{or} \qquad P_n(x)\sin(\gamma x)$$

where P_n is a polynomial of degree n, we look for a particular integral having the form

$$Q_n(x)\cos(\gamma x) + R_n(x)\sin(\gamma x),$$

where Q_n and R_n are polynomials of degree n.

Note that in the case $\cos(\gamma x)$ or $\sin(\gamma x)$ is a solution to the homogeneous differential equation, we have to look for a particular integral having the form

$$Q_{n+1}(x)\cos(x) + R_{n+1}(x)\sin(x),$$

where Q_{n+1} and R_{n+1} are polynomials of degree $n+1$.

Khan Academy $y'' - 3y' - 4y = 2\sin x$

- Example:

$$y'' - 5y' + 6y = 100\sin(4x)$$

The general solution of the homogeneous equation is given by:

$$y = \alpha e^{2x} + \beta e^{3x}.$$

Then we look for a particular integral

$$y_I = \mu\sin(4x) + \nu\cos(4x).$$

Since

$$y_I' = 4\mu\cos(4x) - 4\nu\sin(4x),$$

and

$$y_I'' = -16\mu\sin(4x) - 16\nu\cos(4x),$$

we have

$$10(-\mu + 2\nu)\sin(4x) - 10(2\mu + \nu)\cos(4x) = 100\sin(4x),$$

and then It follows that:

$$\begin{cases} 10(-\mu + 2\nu) = 100 \\ -10(2\mu + \nu) = 0 \end{cases}$$

Finally we get:

$$\begin{cases} \mu = -2 \\ \nu = 4 \end{cases}$$

The general solution to the integral equation is then given by:

$$y = -2\sin(4x) + 4\cos(4x) + \alpha e^{2x} + \beta e^{3x}, \qquad \alpha, \beta \in \mathbb{R}.$$

3.2.4 Case of a more general right-hand

In the case of a right-hand side having the expression

$$P_n(x)e^{\gamma x}\sin(\mu x) + Q_n(x)e^{\gamma x}\cos(\mu x),$$

where P_n and Q_n are polynomials of degree n, we look for a particular integral having the form

$$y_I = R_n(x)e^{\gamma x}\sin(\mu x) + S_n(x)e^{\gamma x}\cos(\mu x),$$

where R_n and S_n are polynomials of degree n.

3.2.5 Case of a right-hand side as a sum of previous functions

In this case, we look for a particular integral for each function in the sum. The particular integral for the differential equation is then equal to the sum of the particular integrals.

Khan Academy
$y'' - 3y' - 4y =$
$3e^{2x} + 2\sin x + 4x^2$.