Integration

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1 The indefinite integral

1.1 Definition

Let f be a function defined from $\mathbb{R}$ into $\mathbb{R}$. Then we can define the function g as the derivative of f:

$$g(x) = f'(x), \quad \forall x \in \mathbb{R}.$$

This defines a transform which to any function associates another function as its derivative.

The *inverse* transform would associate to a function f a function F such that the derivative of the latter function is f, i.e.

$$F'(x) = f(x), \quad \forall x \in \mathbb{R}.$$

This transform is called the anti derivative transform and F is called the anti derivative of f and is usually denoted by:

$$F = \int f(x)\,\mathrm{d}x.$$

1.2 Examples

$F(x) = \int x^n \,\mathrm{d}x$:
The function F has to satisfy $F'(x) = x^n$. If we differentiate ax^m, we get amx^{m-1}. Then, to get the answer, one has to find a and m such that $m-1 =$

n and $am = 1$. Then, $m = n + 1$ and $a(n + 1) = 1$. Here we have to face 2 cases: if $n \neq -1$, then $a = \frac{1}{n+1}$ and $F(x) = \frac{1}{n+1}x^{n+1}$ is a solution. If $n = -1$, then we cannot find a such that $a(n+1) = 1$ because $n+1 = 0$. In that case the solution is given by the ln function:

$$\int \frac{1}{x}\,\mathrm{d}x = \ln(x),$$

because $\frac{\mathrm{d}\ln(x)}{\mathrm{d}x} = \frac{1}{x}$.

We can remark that if F is an anti-derivative of f, then for any constant $c \in \mathbb{R}$, $F + c$ is an anti-derivative of f. In fact, if $\frac{\mathrm{d}F(x)}{\mathrm{d}x} = f(x)$, then $\frac{\mathrm{d}F(x)+c}{\mathrm{d}x} = f(x)$. The anti-derivative, or indefinite integral, is defined up to an additive constant. Therefore, we can write:

$$\frac{\mathrm{d}\ln(x)}{\mathrm{d}x} = \frac{1}{x} + c, \qquad c \in \mathbb{R}.$$

$\int \sin(x)\,\mathrm{d}x$.
Since $\frac{\mathrm{d}(-\cos(x))}{\mathrm{d}x} = \sin(x)$, we have:

$$\int \sin(x)\,\mathrm{d}x = -cos(x) + c, \qquad c \in \mathbb{R}.$$

$\int e^x \mathrm{d}x$:
Since $\frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x$, we have:

$$\int e^x dx = e^x + c, \qquad c \in \mathbb{R}.$$

$\int e^{ax} dx$:
From $\frac{d\alpha e^{\beta x}}{dx} = \alpha\beta e^{\beta x}$, we deduce that in order to get e^{ax} on the right-hand side, we have to satisfy $\alpha\beta = 1$ and $\beta = a$, i.e. $\alpha = \frac{1}{a}$. Then we have:

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + c, \qquad c \in \mathbb{R}.$$

$\int \cos(ax) dx$:
From the previous examples, we can deduce that:

$$\int \cos(ax) dx = \frac{\sin(ax)}{a} + c, \qquad c \in \mathbb{R}.$$

In general, all indefinite integrals of elementary functions can be treated in a similar way and can be found in the tables.

1.3 Properties

The indefinite integral is a linear transform, which means that for any functions f and g and any real constant α, the following properties hold:

(i)
$$\int \alpha f(x)dx = \alpha \int f(x)dx.$$

(ii)
$$\int \big(f(x) + g(x)\big)dx = \int f(x)dx + \int g(x)dx.$$

This property is very useful, since it allows us to determine easily indefinite integrals of linear combinations of elementary functions.

$$\begin{aligned}\int \left(3x^5 - 7x^2 + e^{2x} - \frac{3}{x}\right)dx =&3 \int x^5dx - 7\int x^2dx + \int e^{2x}dx - 3\int \frac{1}{x}dx \\ =&\frac{3}{6}x^6 - \frac{7}{3}x^3 + \frac{1}{2}e^{2x} - 3\ln(x)\end{aligned}$$

Remark 1. *The above properties apply only for linear combinations of functions (sums and multiplication by constants). In any case the following integral:*

$$\int e^x \sin(x)dx$$

can be transformed into

$$e^x \int \sin(x)dx, \quad \textit{or} \quad \int e^x dx \int \sin(x)dx.$$

This is COMPLETELY FALSE!!

2 Usual methods of integration

2.1 Integration by substitution

Many integrals can be solved using an appropriate substitution. There is no universal rule to determine whether we should use a substitution and which substitution should be used.

We can see how the substitution method works with the following examples:

(i)

$$I = \int \frac{dx}{\sqrt{a^2 - x^2}},$$

where a is some real constant.

Here we introduce the variable u as:

$$x = a\sin(u).$$

Then

$$\frac{dx}{du} = a\cos(u),$$

and

$$dx = a\cos(u)du.$$

We also have:

$$\begin{aligned}\sqrt{a^2 - x^2} =& \sqrt{a^2 - a^2\sin(u)} \\ =& \sqrt{a^2\cos^2(u)} \\ =& |a\cos(u)|.\end{aligned}$$

If, in addition, we suppose that $a > 0$, we can prove that $\cos(u) > 0$, and then:

$$\sqrt{a^2 - x^2} = a\cos(u).$$

Finally, we get:

$$\begin{aligned}I =& \int \frac{a\cos(u)du}{a\cos(u)} \\ =& \int du \\ =& u \\ =& \sin^{-1}\left(\frac{x}{a}\right) + c\end{aligned}$$

(ii)

$$I = \int \frac{dx}{\sqrt{2 + 3x}}.$$

Here we introduce the variable u as:

$$u = 2 + 3x.$$

Then:

$$du = 3dx,$$

or

$$dx = \frac{1}{3}du.$$

So,

$$\begin{aligned}
I &= \int \frac{\frac{1}{3}du}{\sqrt{u}} \\
&= \frac{1}{3}\int u^{-\frac{1}{2}}du + c \\
&= \frac{1}{3}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\
&= \frac{2}{3}\sqrt{2+3x} + c.
\end{aligned}$$

(iii)

$$I = \int \frac{\cos(\theta)}{\sin^3(\theta)}d\theta.$$

Here we introduce the variable u as $u = \sin(\theta)$, then we have:

$$du = \cos(theta)d\theta,$$

and

$$\begin{aligned}
I &= \int \frac{du}{u^3} \\
&= \frac{1}{-2}u^{-2} + c \\
&= -\frac{1}{2\sin^2(\theta)} + c.
\end{aligned}$$

A shortened version of this is:

$$\begin{aligned}
I &= \int \frac{d\big(sin(\theta)\big)}{\sin^3(\theta)} \\
&= \int \sin^{-3}(\theta)d\big(\sin(\theta)\big) \\
&= \frac{1}{-2}\sin^{-2}(\theta) + c \\
&= -\frac{1}{2\sin^2(\theta)}.
\end{aligned}$$

2.2 Standard integrals

Integrals of standard functions can be found on tables, but in many cases, one has to transform the integral into a standard form before applying the table results.

A standard result is the following:

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right).$$

The following example shows how we transform the integral $I = \int \frac{dx}{5x^2+7}$ into a standard form in order to apply the previous result.

$$\begin{aligned} I =& \int \frac{dx}{5x^2 + 7} \\ =& \frac{1}{5} \int \frac{1}{x^2 + \frac{7}{5}} \\ =& \frac{1}{5} \int \frac{1}{x^2 + \left(\sqrt{\frac{7}{5}}\right)^2} \\ =& \frac{1}{5} \frac{1}{\sqrt{\frac{7}{5}}} \tan^{-1}\left(\frac{x}{\sqrt{\frac{7}{5}}}\right) \\ =& \frac{1}{\sqrt{35}} \tan^{-1}\left(\sqrt{\frac{5}{7}} x\right) \end{aligned}$$

2.3 Trigonometric fractions

In the case of integrals of polynomials and fractions of trigonometric functions of the variable x, the substitution by

$$t = \tan\left(\frac{x}{2}\right)$$

can sometimes give the solution.

With this substitution, we have:

$$\tan(x) = \frac{2\tan\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)} = \frac{2t}{1 - t^2},$$

$$\sin(x) = \frac{2t}{1 + t^2},$$

$$\cos(x) = \frac{1-t^2}{1+t^2},$$

and

$$\frac{dt}{dx} = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right),$$

which means that

$$dx = \frac{2dt}{1+t^2}.$$

We can use this method to solve $I = \int \frac{dx}{1+2\cos(x)}$. If we define t as

$$t = \tan\frac{x}{2},$$

then, using the previous formulas, we get

$$\begin{aligned}
I =& \int \frac{2dt}{(1+t^2)\left(1+2\frac{1-t^2}{1+t^2}\right)} \\
=& \int \frac{2dt}{3-t^2} \\
=& 2\int \frac{dt}{(\sqrt{3})^2 - t^2} && \text{this is in standard form, see table} \\
=& 2\frac{1}{\sqrt{3}}\tanh^{-1}\left(\frac{t}{\sqrt{3}}\right) + c \\
=& \frac{2}{\sqrt{3}}\tanh^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) + c
\end{aligned}$$

Alternatively, we can write:

$$\begin{aligned}
\frac{2}{(\sqrt{3})^2 - t^2} =& \frac{2}{(\sqrt{3}-t)(\sqrt{3}-t)} \\
=& \frac{\frac{1}{\sqrt{3}}}{\sqrt{3}-t} + \frac{\frac{1}{\sqrt{3}}}{\sqrt{3}+t}.
\end{aligned}$$

Then,

$$\begin{aligned}
I =& \int \frac{\frac{1}{\sqrt{3}}}{\sqrt{3}-t}dt + \int \frac{\frac{1}{\sqrt{3}}}{\sqrt{3}+t}dt \\
=& -\frac{1}{\sqrt{3}}\ln(\sqrt{3}-t) + \frac{1}{\sqrt{3}}\ln(\sqrt{3}+t) \\
=& \frac{1}{\sqrt{3}}\ln\left(\frac{\sqrt{3}+t}{\sqrt{3}-t}\right)
\end{aligned}$$

2.4 Completing the square

This method consists in completing $ax^2 + bx$ by the right constant in order to obtain the square formula

$$a\left(x + \frac{b}{2a}\right)^2.$$

Suppose we have to solve the following integral:

$$I = \int \frac{dx}{2x^2 + 2x + 3},$$

then

$$\begin{aligned} I =& \frac{1}{2}\int \frac{dx}{x^2 + x + \frac{3}{2}} \\ =& \frac{1}{2}\int \frac{dx}{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{3}{4}} \\ =& \frac{1}{2}\int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \frac{5}{4}} \end{aligned}$$

Now, if we introduce u by $u = x + \frac{1}{2}$, then

$$\begin{aligned} I =& \frac{1}{2}\int \frac{du}{u^2 + \left(\frac{\sqrt{5}}{2}\right)^2} && \text{this is a standard form} \\ =& \frac{1}{2}\frac{1}{\frac{\sqrt{5}}{2}}\tan^{-1}\frac{u}{\frac{\sqrt{5}}{2}} \\ =& \frac{1}{\sqrt{5}}\tan^{-1}\frac{2x+1}{\sqrt{5}} \end{aligned}$$

2.5 Integration by parts

If u and v are two functions, then the derivative of the product uv is given by the following formula:

$$(uv)' = u'v + uv'.$$

Now, we integrate this identity, because of the linearity of the integral, we get:

$$\int (uv)'dx = \int u'vdx + \int uv'dx.$$

Suppose now that we have to solve the integral of a particular function f that can be written as a product of a function u and the derivative of a function v, i.e.

$$f = uv'.$$

Then, we can easily deduce from the previous formula that:

$$\begin{aligned}\int f dx = &\int uv' dx\\ =&\int (uv)' dx - \int u'v dx\\ =&uv - \int u'v dx\end{aligned}$$

This formula can be summarized as follows:

$$\int u dv = uv - \int v du.$$

Example 1

To solve

$$I = \int x \cos x dx,$$

we define $u = x$ and $\cos x dx = dv$. Then, we have $du = dx$ and $v = \int \cos x dx = \sin x$. Now if we substitute these values in the previous formula, we get:

$$\begin{aligned}I =&x \sin x - \int \sin x dx\\ =&x \sin x + \cos x\end{aligned}$$

Example 2

Sometimes, we need integrate by parts the new integral as in the following example:

$$J = \int x^2 \sin x dx$$

Define $u = x^2$ and $\sin x dx = dv$. Then, we have $du = 2xdx$ and $v = \int \sin x dx = -\cos x$. Then:

$$\begin{aligned} J =& -x^2 \cos x - \int 2x(-\cos x)dx \\ =& -x^2 \cos x + 2\int x \cos x dx \\ =& -x^2 \cos x + 2I \\ =& -x^2 \cos x + 2(x \sin x + \cos x) \\ =& (-x^2 + 2) \cos x + 2x \sin x \end{aligned}$$

After the first integration by parts, we arrived to the result $J = x^2 \cos x + 2\int x \cos x dx$, then we needed to do another integration by parts to solve $\int x \cos x dx$ (which has been done in the previous example).

Example 3

In other cases, integration by parts leads back to the original integral. This can allow to solve the integral by solving a simple algebraic equation.

$$K = \int e^x \sin x dx.$$

Define $u = e^x$ and $dv = \sin x dx$. It follows that $du = e^x dx$ and $v = -\cos x$. Then:

$$\begin{aligned} K =& -e^x \cos x - \int (-\cos x)e^x dx \\ =& -e^x \cos x + \int \cos x e^x dx. \end{aligned}$$

Now define again $u = e^x$ and $dv = \cos x dx$. It follows that $du = e^x dx$ and $v = \sin x$. Then:

$$\begin{aligned} \int \cos x e^x dx =& e^x \sin x - \int e^x \sin x dx \\ =& e^x \sin x - K \end{aligned}$$

Now substituting this in the previous equations we get the following algebraic equation:

$$K = -e^x \cos x + e^x \sin x - K,$$

or equivalently

$$2K = -e^x \cos x + e^x \sin x,$$

which means that:

$$K = \frac{1}{2}e^x(\sin x - \cos x).$$

3 The definite integral

3.1 Definition

Definition 1. *Let $a, b \in \mathbb{R}$ such that $a \leq b$ and let f be an integrable function over $[a, b]$. Suppose that $F = \int f(x)dx$, i.e. $\frac{dF(x)}{dx} = f(x)$.*

The definite integral of f between a and b is denoted by $\int_a^b f(x)dx$ and is given by:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

In the following example we compute the integral of $f(x) = x^2 + 3x - 5$ between -1 and 4.

$$\begin{aligned}
\int_{-1}^4 f(x)dx &= \int_{-1}^4 (x^2 + 3x - 5)dx \\
&= \left[\frac{1}{3}x^3 + \frac{3}{2}x^2 - 5x\right]_{-1}^4 \\
&= \left(\frac{1}{3}(4)^3 + \frac{3}{2}(4)^2 - 5(4)\right) - \left(\frac{1}{3}(-1)^3 + \frac{3}{2}(-1)^2 - 5(-1)\right) \\
&= \frac{115}{6}
\end{aligned}$$

3.2 Properties

(i) Suppose that for any $x \in [a, b]$, $f(x) \geq 0$, then:

$$\int_a^b f(x)dx \geq 0.$$

(ii) For any $a, b \in \mathbb{R}$, we have:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

(iii) for any $a, b, c \in \mathbb{R}$, we have:

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

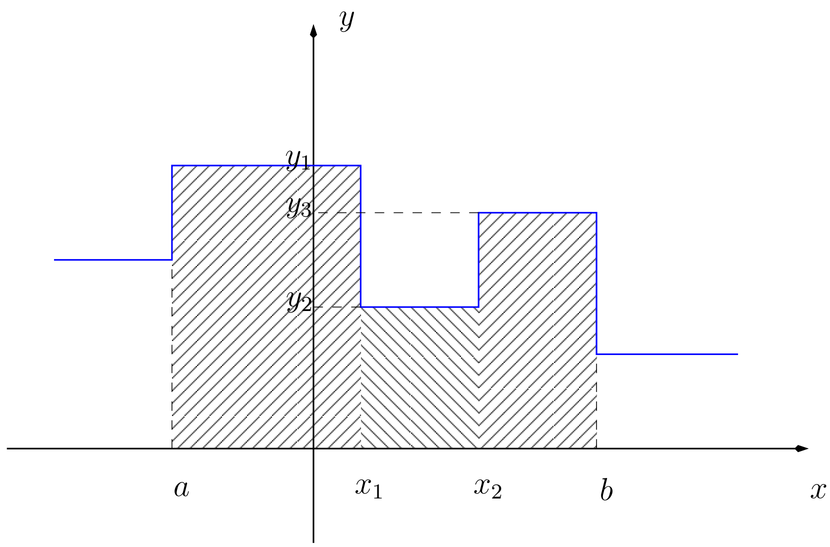

Figure 1: Integration of a piecewise constant function.

3.3 Interpretation

Suppose we have to integrate a piecewise constant function between a and b as shown on Figure 1.

Since:

$$f(x) = \begin{cases} y_1 & \forall x \in (a, x_1) \\ y_2 & \forall x \in (x_1, x_2) \\ y_3 & \forall x \in (x_2, x_3) \end{cases}$$

we can deduce from property (iii) that:

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \int_{x_2}^{x_3} f(x)dx \\ &= \int_a^{x_1} y_1 dx + \int_{x_1}^{x_2} y_2 dx + \int_{x_2}^{x_3} y_3 dx \\ &= y_1(x_1 - a) + y_2(x_2 - x_1) + y_3(b - x_2) \end{aligned}$$

We see clearly that in the case of a piecewise constant function f, the integral $\int_a^b f(x)dx$ corresponds to the area of the surface located between the curve of f, the x axis and the vertical lines passing at a and b.

Now suppose we want to determine the area of the surface located between the curve of f, the x axis and the vertical lines passing at a and b for a more general positive function on $[a, b]$.

If we divide the interval $[a, b]$ into 10 subintervals $[x_n, x_{n+1}]$, $0 \leq n \leq 9$, of equal lengths $\frac{b-a}{10}$, where $x_0 = a$ and $x_{10} = b$, we can approximate the area of the surface by the area of the rectangles of sides $[x_n, x_{n+1}]$ and $[0, f(x_n)]$ as shown on Figure 2.

This approximation can be improved if we divide $[a, b]$ into $N > 10$ intervals of length $\frac{b-a}{N}$. The higher is N, the better the approximation will be.

If N increases indefinitely, we will converge to the area of the original surface which can be interpreted as the sum of areas of an infinity of rectangles of infinitesimal (infinitely small) width and of height $f(x)$, having an infinitesimal surface $f(x)dx$.

The integral $\int_a^b f(x)dx$ corresponds then to the area of the surface located between the curve of f, the x axis and the vertical lines passing at a and b.

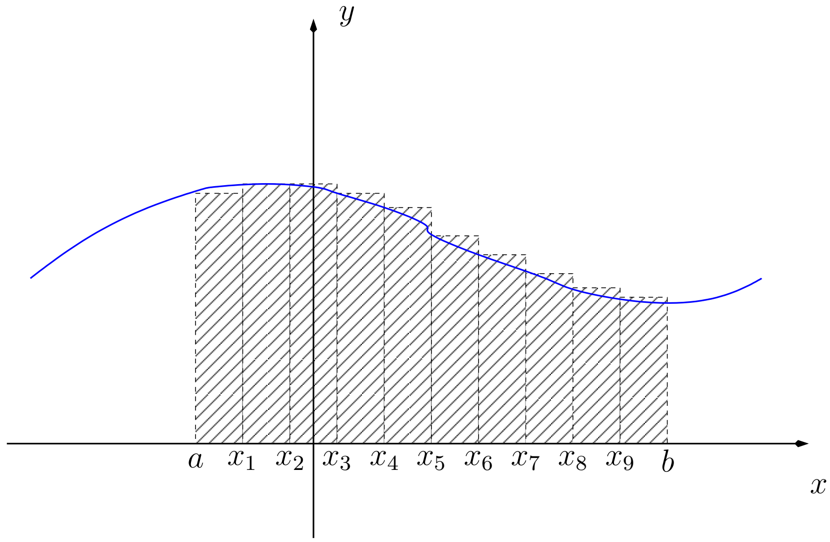

Figure 2: Integration of a more general positive function.

Suppose now that the function f satisfies the following condition:

$$\begin{cases} f(x) \leq 0 & \forall x \in (a, c) \\ f(x) \geq 0 & \forall x \in (c, b) \end{cases}$$

as shown on Figure 3.

Then using property (iii), we have:

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx \\ &= - \int_a^c -f(x)dx + \int_c^b f(x)dx \end{aligned}$$

Since the curve of $-f$ is the symmetric of the one of f about the x axis, $\int_a^c -f(x)dx$ is equal to the area of the surface located between the curve of f, the x axis and the vertical lines passing at a and c. (because $-f$ is positive)

We can now generalize the interpretation of the integral to general functions as follows : $\int_a^b f(x)dx$ corresponds to the algebraic area of the surface located between the curve of f, the x axis and the vertical lines passing at a and b. The adjective algebraic means that the surface will be counted positively if the function is positive and negatively if the function is negative.

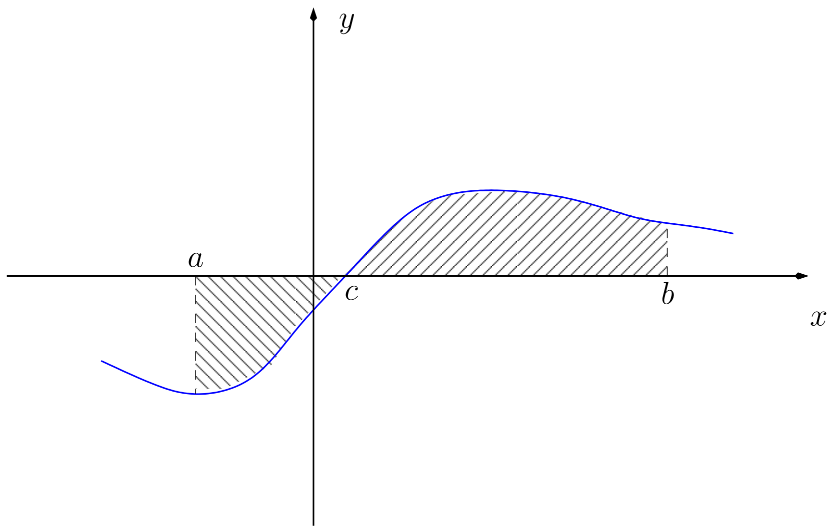

Figure 3: Integration of a general function.

3.4 Applications

3.4.1 Find the area between two curves

Let A be the area of the surface between the curves of two functions f and g for $x \in [a, b]$, then:

$$A = \int_a^b |f(x) - g(x)|dx.$$

In fact, we can see in the example of Figure 4 that:

$$\begin{aligned}
A =& A_1 + A_2 \\
=& \int_a^c (f(x) - g(x))dx + \int_c^b (g(x) - f(x))dx \\
=& \int_a^c |f(x) - g(x)|dx + \int_c^b |g(x) - f(x)|dx \\
=& \int_a^b |f(x) - g(x)|dx
\end{aligned}$$

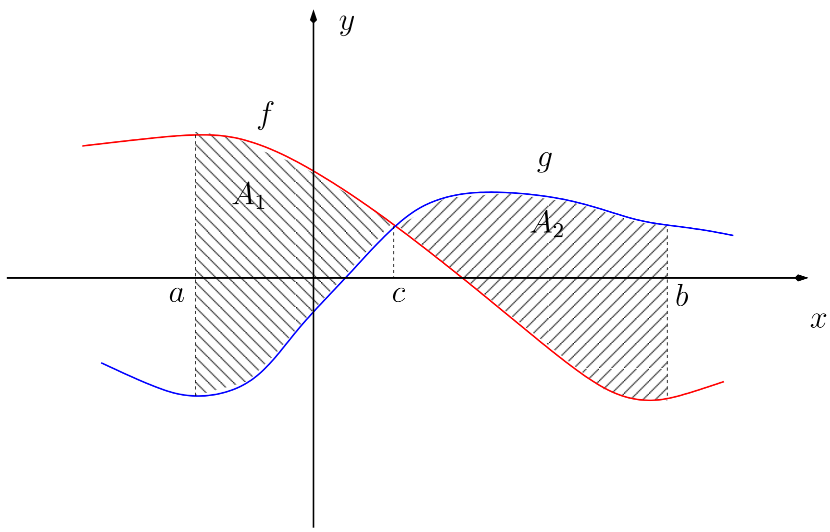

Figure 4: Area between 2 curves.

3.4.2 Volumes of revolution

Suppose we want to determine the volume delimited by the surface obtained when rotating the curve of a function f about the x axis for $x \in [a, b]$. Then we can divide this volume into infinitesimal cylinders around $(x, 0)$ of radius $f(x)$ and height dx.

The volume V is then obtained by summing the infinitesimal volumes $dv = \pi f(x)^2 dx$:

$$V = \int_a^b \pi f(x)^2 dx.$$

This is shown diagrammatically on Figure 5.

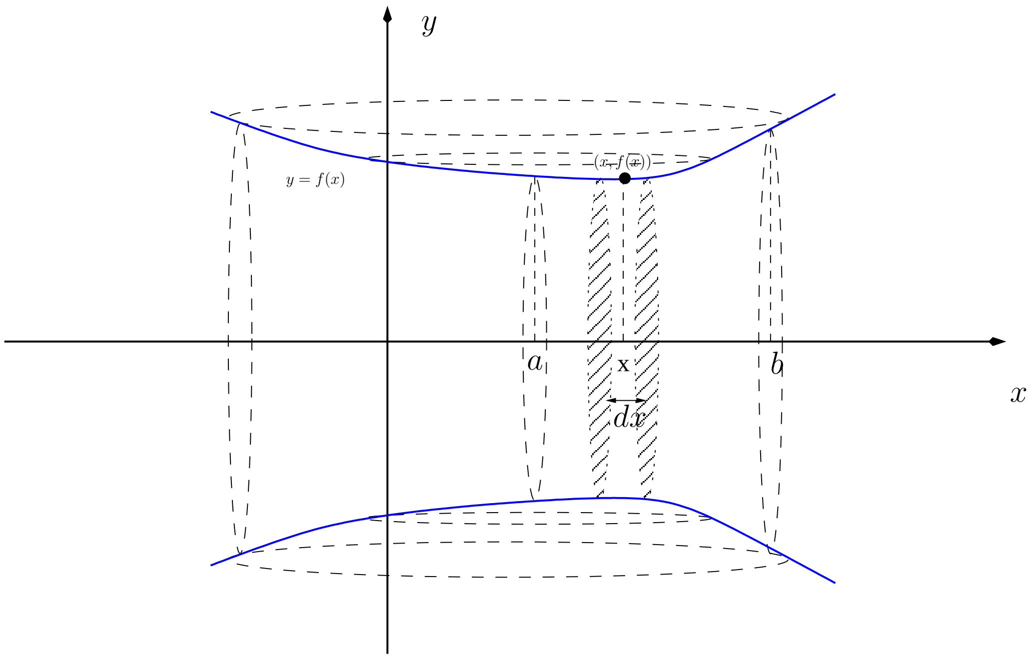

Figure 5: Volume of revolution.

Example: to find the volume to a cone of height h and radius R, we can consider it as the product of the revolution of the curve $y = f(x) = \frac{R}{h}x$ about the x axis, the basis being at $x = h$. Then following the previous formula, we have:

$$\begin{aligned} V &= \int_0^h \pi f(x)^2 dx \\ &= \int_0^h \pi \left(\frac{R}{h}x\right)^2 dx \\ &= \pi \frac{R^2}{h^2}\left[\frac{x^3}{3}\right]_0^h \\ &= \frac{\pi R^2 h}{3} \end{aligned}$$

3.4.3 Length of a curve

Consider a function f and suppose we have to determine the length of the curve of f between $x = a$ and $x = b$. We can approach this curve by N straight lines $[P_0, P_1]$, $[P_1, P_2]$,..., $[P_N, P_{N+1}]$ as shown on figure 6.

Suppose that the points $(x_j)_{0 \le j \le N+1}$ are equidistant, i.e. $x_{j+1} - x_j = \frac{b-a}{N} = \Delta x$. Then, since the point P_j has coordinates $(x_j, f(x_j))$, the length

δl_j of $[P_j, P_{j+1}]$ is given by:

$$\begin{aligned} \delta l_j^2 =& (x_{j+1} - x_j)^2 + (f(x_{j+1}) - f(x_j))^2 \\ =& (\Delta x)^2 + \Delta f(x_j)^2 \end{aligned}$$

Now suppose that the number of subdivisions N goes to infinity, then the sum of the lengths of $[P_j, P_{j+1}]$ will converge to the length of the curve.

But if the N goes to infinity, we will obtain an infinity of straight lines of infinitesimal length dl given by:

$$\begin{aligned} dl^2 =& (dx)^2 + df(x_)^2 \\ =& (dx)^2 + \left(\frac{df}{dx}\right)^2 dx^2 \\ =& \left(1 + f'(x)^2\right) dx^2 \end{aligned}$$

and

$$ds = \sqrt{1 + f'(x)^2} dx$$

Since the length of the curve is given by $L = \int_a^b dl$, we have:

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx.$$

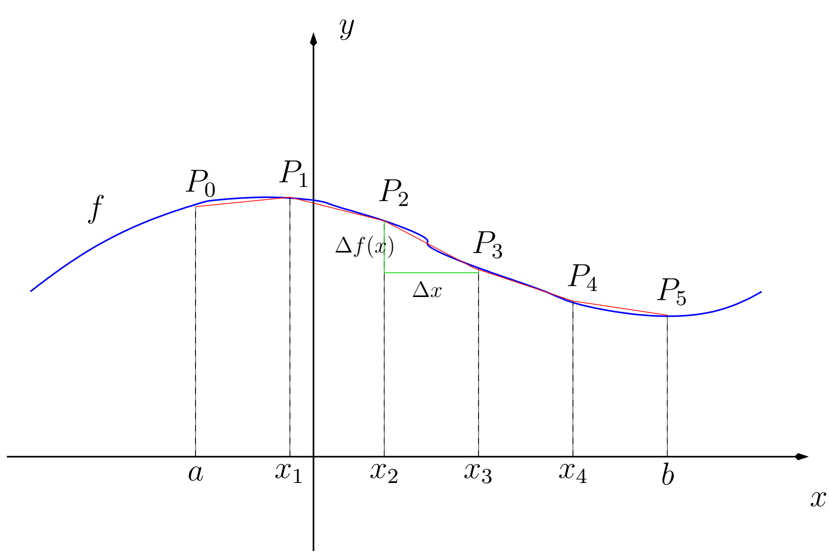

Figure 6: Length of a curve.

We can use this formula for finding the circumference of a circle of radius R. The equation of the circle of center O and radius R is:

$$x^2 + y^2 = R^2.$$

The circle can be considered as made of two halfs (one in the half-plane $y \geq 0$ and one in the half-plane $y \leq 0$). The circumference is then twice the length of a half-circle.

The half-circle lying in the half-plane $y \geq 0$ is the curve of the function $f(x) = \sqrt{R^2 - x^2}$. To find the derivative of f we can derive the following equations:

$$x^2 + f(x)^2 = R^2,$$

which gives:

$$2x + 2f(x)f'(x) = 0,$$

and then

$$f'(x) = -\frac{x}{f(x)} = -\frac{x}{\sqrt{R^2 - x^2}}.$$

It follows that the circumference of the circle is :

$$\begin{aligned}
S =& 2\int_{-R}^{+R} \sqrt{1 + f'(x)^2} dx \\
=& 2\int_{-R}^{+R} \sqrt{1 + \frac{x^2}{R^2 - x^2}} dx \\
=& 2R\int_{-R}^{+R} \frac{1}{\sqrt{R^2 - x^2}} dx \\
=& 2R\left[\sin^{-1}\left(\frac{x}{R}\right)\right]_{-R}^{+R} \\
=& 2R\left(\sin^{-1}(1) - \sin^{-1}(-1)\right) \\
=& 2R\left(\frac{\pi}{2} - (-\frac{\pi}{2})\right) \\
=& 2\pi R.
\end{aligned}$$

3.4.4 Centroid of area under a curve

For a set of point masses $(m_i)_{1 \leq i \leq N}$ located at points $(x_i, y_i)_{1 \leq i \leq N}$, the centroid has coordinates $(\bar{x}, \bar{y})$ given by:

$$\begin{cases} \bar{x} = \frac{\sum_i m_i x_i}{\sum_i m_i} \\ \bar{y} = \frac{\sum_i m_i y_i}{\sum_i m_i} \end{cases}$$

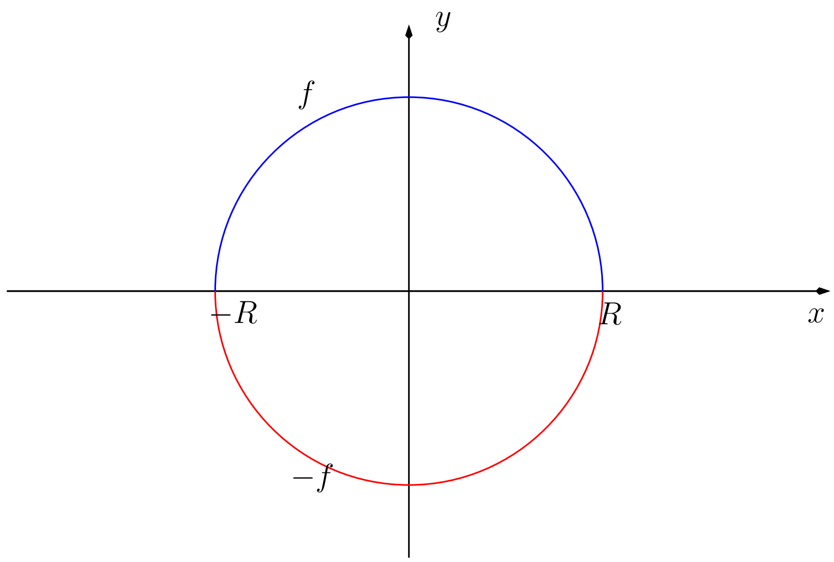

Figure 7: Circumference of a circle.

Consider now the area under the curve of a function f for $x \in [a, b]$. The coordinates of the centroid of this area can be obtained by considering this area as an infinite sum of point masses $dxdy$ at points (x, y), $a \leq x \leq b$, $0 \leq y \leq f(x)$.

In order to find the centroid we can even divide the area into infinitesimal rectangles of width dx and height $f(x)$ centered around x which has an infinitesimal mass $dM = |f(x)|dx$ and which centroid is located at $\left(x, \frac{f(x)}{2}\right)$.

Then, the coordinates of the centroid of the area are given by:

$$\begin{cases} \bar{x} = \dfrac{1}{A}\displaystyle\int_a^b x dM \\ \bar{y} = \dfrac{1}{A}\displaystyle\int_a^b \dfrac{f(x)}{2} dM \end{cases}$$

where A is the total area:

$$A = \int_a^b f(x)dx.$$

Replacing dM by its expression in terms of x, we get:

$$\begin{cases} \bar{x} = \dfrac{1}{A}\displaystyle\int_a^b x|f(x)|dx \\ \bar{y} = \dfrac{1}{A}\displaystyle\int_a^b \dfrac{f(x)}{2}|f(x)|dx \end{cases}$$

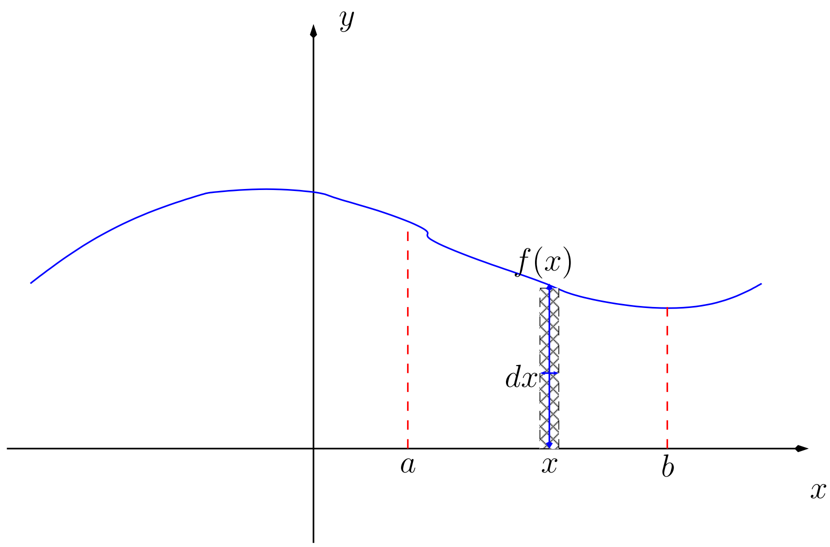

Figure 8: Centroid of an area under a curve.

This formula can be generalized to the area between two curves f and g, where $f(x) \leq g(x)$ for any $x \in [a, b]$. In this case the infinitesimal rectangles will have a width of dx and extend between $f(x)$ and $g(x)$ (instead of 0 and $f(x)$), then their infinitesimal mass is given by $dM = |g(x) - f(x)|dx$ and their centroid is located at $(x, \frac{f(x)+g(x)}{2})$.

Then:

$$\begin{aligned}\bar{x} =& \frac{1}{A}\int_a^b x dM \\ =& \frac{1}{A}\int_a^b x|g(x) - f(x)|dx\end{aligned}$$

and

$$\begin{aligned}\bar{y} =& \frac{1}{A}\int_a^b \frac{f(x)+g(x)}{2} dM \\ =& \frac{1}{A}\int_a^b \frac{f(x)+g(x)}{2}|g(x) - f(x)|dx\end{aligned}$$

3.4.5 Moments of inertia

For a set of point masses $(m_i)_{1\leq i\leq N}$ located at points $(x_i, y_i)_{1\leq i\leq N}$, the moments of inertia I_x about the x axis, I_y about the y, and I_0 about the origin

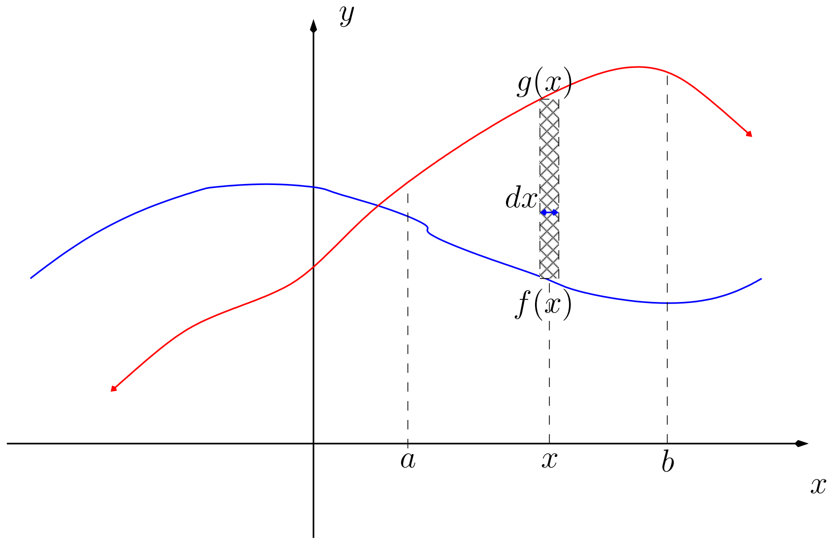

Figure 9: Centroid of the area between 2 curves.

are given by:

$$I_x = \sum_i y_i^2 m_i,$$

$$I_y = \sum_i x_i^2 m_i,$$

$$I_0 = I_x + I_y = \sum_i (y_i^2 + x_i^2) m_i.$$

Now, suppose we have to determine the moment of inertia I_y of an area extending between the curves of two functions $y = f(x)$ and $y = g(x)$, $f(x) \leq g(x)$, as shown in Figure 10. We can divide this area into infinitesimal rectangles of width dx extending between $y = f(x)$ and y=$g(x)$. All points in such a rectangle are at a distance x from the y axis. The total mass of the rectangle is $dM = |g(x) - f(x)|dx$. It follows that:

$$\begin{aligned} I_y &= \int_a^b x^2 dM \\ &= \int_a^b x^2 |g(x) - f(x)| dx \end{aligned}$$

Let us now determine the moment of inertia I_x of the same surface about the x axis. To do so, we can consider an infinitesimal element of surface of width dx and height dy located at a point (x, y) of the surface as shown on Figure 11. (this means that $a \leq x \leq b$ and $f(x) \leq y \leq g(x)$)

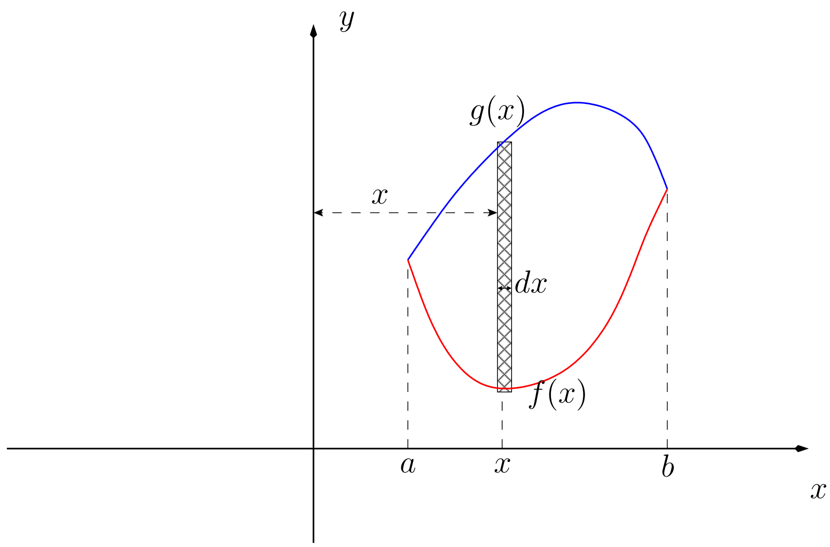

Figure 10: Moment of inertial about y axis.

This element has a mass equal to $d^2M = dxdy$ and is located at a distance y from the x axis. Its moment d^2I_x is then given by:

$$d^2I_x = y^2d^2M = y^2dxdy.$$

Now consider the infinitesimal strip of width dx located at x and extending from $f(x)$ to $g(x)$. The moment of inertial of this strip about the x axis is given by summing the moments of inertia of all elements $dxdy$ located at points (x, y) for y satisfying $f(x) \leq y \leq g(x)$. This moment dI_x is given by:

$$\begin{aligned}
dI_x =& \int_{y=f(x)}^{g(x)} d^2I_x \\
=& \int_{y=f(x)}^{g(x)} y^2d^2M \\
=& \int_{y=f(x)}^{g(x)} y^2dxdy \\
=& dx \int_{y=f(x)}^{g(x)} y^2dy \\
=& dx \left[\frac{y^3}{3}\right]_{y=f(x)}^{g(x)} \\
=& \frac{g^3(x) - f^3(x)}{3}dx
\end{aligned}$$

Finally to get the moment I_x of the total surface about the x axis, we have to sum the moments dI_X for $x \in [a, b]$:

$$
\begin{aligned}
I_x &= \int_{x=a}^{b} dI_x \\
&= \int_{x=a}^{b} \frac{g^3(x) - f^3(x)}{3} dx
\end{aligned}
$$

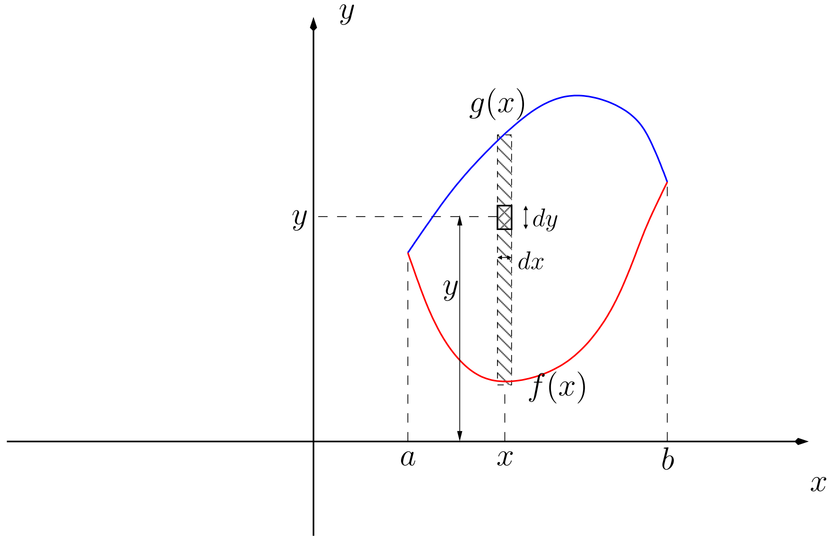

Figure 11: Moment of inertial about x axis.

Suppose now that we consider the moment of inertial I_x about the x axis of an area extending between the curves of two functions $x = h(y)$ and $x = k(y)$, $h(y) \leq k(y)$, as shown in Figure 12. This moment can be found dividing the area into infinitesimal rectangles of height dy extending between $x = h(y)$ and $x = k(y)$. All points in such a rectangle are at a distance y from the x axis. The total mass of the rectangle is $dM = |k(y_) - h(y)|dy$. It follows that:

$$
I_x = \int_a^b y^2 |k(y) - h(y)| dy
$$

We can apply these formula to find the moments of inertial for the area defined by:

$$
x^2 + y^2 \leq 1, \qquad x \geq 0, \quad y \geq 0.
$$

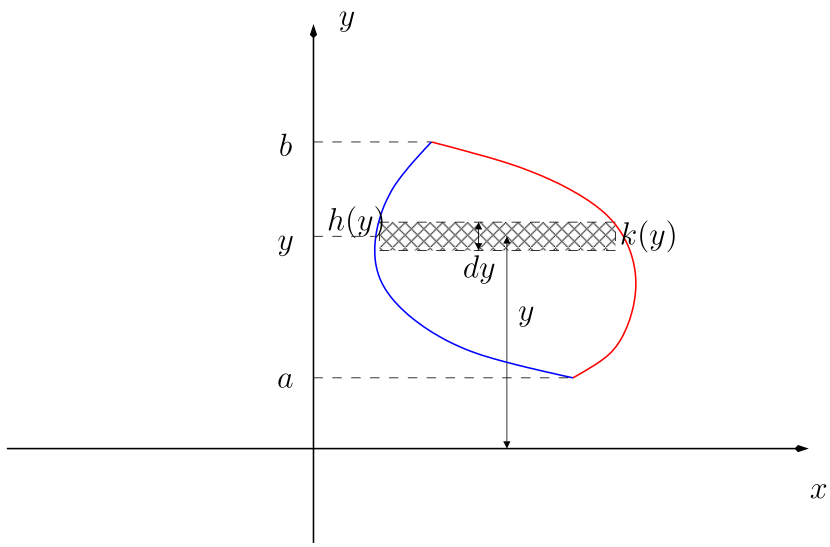

Figure 12: Moment of inertial about x axis.

To find I_y, we set $a = 0$, $b = 1$, $f(x) = 0$, and $g(x) = \sqrt{1-x^2}$. Then

$$I_y = \int_0^1 x^2\sqrt{1-x^2}.$$

Define θ as $x = \sin\theta$, then $dx = \cos\theta d\theta$, and

$$\begin{aligned}
I_y &= \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{1-\sin^2\theta}\cos\theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sin(2\theta)\right)^2 d\theta \\
&= \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1-\cos(4\theta)}{2} d\theta \\
&= \frac{1}{8}\left[\theta - \frac{\sin(4\theta)}{4}\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{8}\left[\frac{\pi}{2}\right] \\
&= \frac{\pi}{16}
\end{aligned}$$

Because of the symmetry of the shape, $I_x = I_y$.

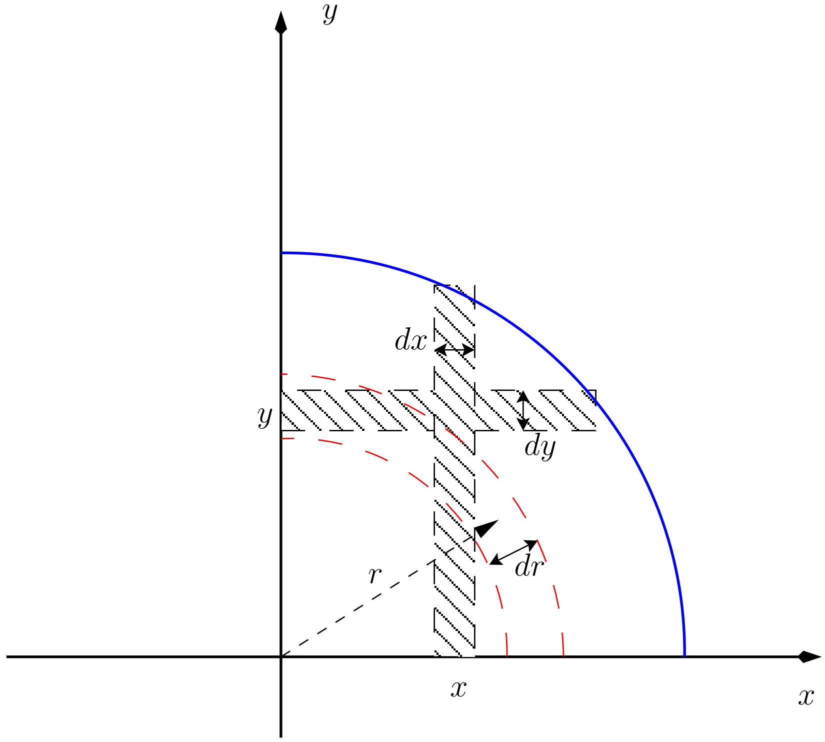

Figure 13: Moment of inertial of a quarter circle.

Definition 2. *If the total mass of an area is M and its moment of inertial is I, then we define the* radius of gyration k *about the corresponding axis by:*

$$k^2 = \frac{I}{M}.$$

Proposition 1. *Let k_x, ky, and k_o be the radius of gyration about the x axis, the y axis, and the origin respectively. Then:*

$$k_o^2 = k_x^2 + k_y^2.$$

This result is a direct consequence of the fact that:

$$I_o = I_x + I_y,$$

which is a consequence of the obvious formula:

$$x^2 + y^2 = r^2.$$

In fact, in the previous example, we can solve directly I_o by dividing the are into rings of infinitesimal width dr having a mass $dM = \frac{2\pi r}{4}dr = \frac{\pi r}{2}dr$, and were each point is at a distance r from the origin. It follows that;

$$\begin{aligned} I_0 &= \int_0^r r^2 dM \\ &= \int_0^r r^2 \frac{\pi r}{2} dr \\ &= \int_0^r \frac{\pi r^3}{2} dr \\ &= \frac{\pi}{2} \left[\frac{r^4}{4} \right]_0^1 \\ &= \frac{\pi}{8} \end{aligned}$$

We can then check that

$$I_x + I_y = \frac{\pi}{16} + \frac{\pi}{16} = \frac{\pi}{8} = I_o.$$

This relation can be useful to find one moment when the two other moments are known.

4 Numerical methods of integration

In many situations, we deal with integrals of complicated functions for which it is impossible to find an explicit expression. To solve such integrals, we use some numerical methods which allow us to find an approximate value of the integral. These methods can be implemented on computers and their precision depends on the discretization parameters. As the discretization increases, the precision is improved, but at the same time, the need of time and memory increases.

4.1 The trapezoidal rule

Suppose we have to find the integral between a and b of some function f. Then we divide the interval $[a, b]$ into N subintervals of equal length $h = \frac{b-a}{N}$: $[x_n, x_{n+1}]$, $0 \leq n \leq N-1$, where $x_n = a + n\frac{b-a}{N} = a + nh$, $0 \leq n \leq N$.

The integral $I = \int_a^b f(x)dx$ can then be approximated by I_N given by:

$$I_N = \frac{h}{2}\Big(f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{N-2}) + 2f(x_{N-1}) + f(x_N) \Big).$$

This formula comes from the fact that we approximate the area under the curve by the sum of the areas of right trapezia of height $[x_n, x_{n+1}]$ and for which the parallel sides have lengths $f(x_n)$ and $f(x_{n+1})$. Its area is then equal to $\frac{h}{2}(f(x_{n+1}) + f(x_n))$.

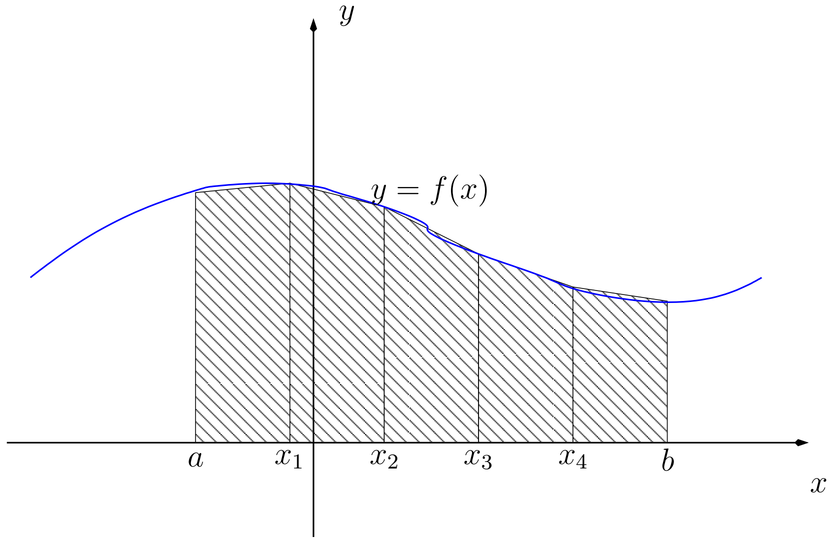

Figure 14: The trapezoidal rule.

4.1.1 Example

We can give an approximation of $I = \int_1^5 \frac{dx}{\sqrt{1+x^3}}$ using 5 strips. We first define $x_0 = 1$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$, $x_5 = 5$. Then we compute I_5 as follows:

$$I_5 = \frac{1}{2}\Bigg(f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5) \Bigg).$$

We find $I_5 \simeq 1.8975$.

4.2 The Simpson's rule

The Simpson's rule is another method for numerical integration. Its justification is more complicated. However, for the same number of subdivisions, the Simpson's rule give a more accurate result than the trapezoidal rule.

The Simpson's rule needs an even number of subdivisions $2N$ of the interval of integration $[a, b]$, defining intervals of width $h = \frac{b-a}{2N}$ which edges are the points $(x_n)_{0 \leq n \leq 2N}$, $x_n = a + \frac{b-a}{2N} n$. We have then $x_0 = a$ and $x_{2N} = b$.

The approximation of $\int_a^b f(x)dx$ given by the Simpson's rule I_N is defined by:

$$\begin{aligned}I_{2N} =&\frac{h}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{2n-1}) \\ &+ 4f(x_{2n}) + 2f(x_{2n+1}) + \ldots + 2f(x_{2N-2}) + 4f(x_{2N-1}) + f(x_{2N})\Big).\end{aligned}$$

4.2.1 Example

Using the Simpson's rule for finding an approximation of $I = \int_1^5 \frac{dx}{\sqrt{1+x^3}}$ with 10 subdivisions, we get:

$$\begin{aligned}I_{10} =&\frac{1}{3}\Big(f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + 2f(4) \\ &+ 4f(4.5) + 2f(5) + 4f(5.5) + 2f(6) + 4f(6.5) + 2f(7) + 4f(7.5) \\ &+ 2f(8) + 4f(8.5) + 2f(9) + 4f(9.5) + f(10)\Big) \\ \simeq&1.1913\end{aligned}$$