Partial Differentiation

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1 Motivation

Partial differentiation is useful in a number of contexts.

Calculating errors Partial differentiation is used to estimate errors in calculated quantities that depend on more than one uncertain experimental measurement.

Thermodynamics Thermodynamic energy functions (enthalpy, Gibbs free energy, Helmholtz free energy) are function of two or more variables. Most thermodynamic quantities (temperature, entropy, heat capacity) can be expressed as derivatives of these functions.

Financial engineering Financial engineers use partial derivatives to assess a portfolio's sensitivity to changes in market conditions (interest rates, volatility). They can hedge against risk by designing portfolios which have zero partial derivative with respect to market values.

Partial differential equations Many laws of nature are best expressed as relations between the partial derivatives of one or more quantities. For instance the Schrödinger equation describes all the laws of chemistry

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi \tag{1}$$

and the Navier-Stokes equation describes all fluid motion

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)\, v = -\frac{1}{\rho}\nabla p + \nu \nabla^2 v \tag{2}$$

where

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \qquad \nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \tag{3}$$

An unfortunate consequence of the generality of these equations is that they are impossible to solve except for in a handful of special cases.

2 Introduction

Definition 1. *A real function of several real variables is a function that depends on more than one variable.*

2.1 Examples

i) $f_1(x, y) = 2x + 3y$ is a function of 2 variables.

ii) $f_2(x, y) = x^2 - 4y^3$ is a function of 2 variables.

iii) $f_3(x, y) = 2xy^2 - 4xe^y$ is a function of 2 variables.

iv) $f_4(x, y, z) = \frac{xy\sqrt{y+z^3}}{x^2+z^2}$ is a function of 3 variables.

3 Partial differentiation

Consider a function of 2 variables $f(x, y)$, then if we keep y fixed ($y = y_0$), we can define a new function of 1 variable g as follows:

$$g(x) = f(x, y_0).$$

Definition 2. *The derivative of g (with respect to x) is called the partial derivative of f with respect to (w.r.t.) the first variable (x) and is denoted by*

$$\frac{\partial f}{\partial x}(x, y_0) = g'(x).$$

If we keep constant x ($x = x_0$), then we can define another new function h as :

$$h(y) = f(x_0, y).$$

The derivative of h (w.r.t. y) is called the partial derivative of f w.r.t. the second variable (y) and is denoted by:

$$\frac{\partial f}{\partial y}(x_0, y) = h'(y).$$

3.1 Examples

i) $f_1(x, y) = 2x + 3y$, $\frac{\partial f_1}{\partial x} = 2 + 0 = 2$, $\frac{\partial f_1}{\partial y} = 0 + 3 = 3$.

ii) $f_2(x, y) = x^2 - 4y^3$, $\frac{\partial f_2}{\partial y} = 2x - 0 = 2x$, $\frac{\partial f_2}{\partial y} = 0 - 12y^2$.

iii) $f_3(x, y) = 2xy^2 - 4xe^y$, $\frac{\partial f_3}{\partial x} = 2y^2 - 4e^y$, $\frac{\partial f_3}{\partial y} = 4xy - 4xe^y$.

3.2 Functions of more than 2 variables

In the case of a function depending on more than 2 variables, the partial derivative w.r.t. one variable is the derivative of this function when we keep constant all other variables.

Consider function $f(x, y, z, t)$ (depending on 4 variables) defined by:

$$f(x, y, z, t) = 3xt^2 \cos(2yz^2).$$

Then we can define 4 partial derivatives:

- $\frac{\partial f}{\partial x} = 3t^2 \cos(2yz^2)$
- $\frac{\partial f}{\partial y} = -6xt^2z^2 \sin(2yz^2)$
- $\frac{\partial f}{\partial z} = -12xyzt^2 \sin(2yz^2)$
- $\frac{\partial f}{\partial t} = 6xt \cos(2yz^2)$

4 Higher order partial derivatives

So far we have seen the first order partial derivatives which are obtained by deriving once a function of several variables w.r.t. one variable.

Suppose we have a function $f(x,y)$, then we can define the function g as the partial derivative of f w.r.t. x:

$$g(x,y) = \frac{\partial f}{\partial x}(x,y).$$

The function g depends on the same variables as f, and we can define its partial derivatives: $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$.

We call the partial derivative of g w.r.t. x "the second-order partial derivative" of f w.r.t. x as it was obtained by deriving f twice w.r.t. x keeping y constant and we denote it by:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}.$$

The partial derivative of g w.r.t y is called the second-order partial derivative of f with respect to x and then to y. It is denoted by:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}.$$

We can define the third-order partial derivatives of f as the partial derivatives of the second-order partial derivatives of f or equivalently as the second-order partial derivatives of the first-order partial derivatives of f.

Remark 1. *For a function depending on 2 variables $f(x,y)$ there exist*

- *2 first-order partial derivative :* $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.
- *4 second-order partial derivatives :* $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y^2}$.
- *8 third-order partial derivatives :* $\frac{\partial^3 f}{\partial x^3}$, $\frac{\partial^3 f}{\partial y \partial x^2}$, $\frac{\partial^3 f}{\partial x \partial y \partial x}$, $\frac{\partial^3 f}{\partial y^2 \partial x}$, $\frac{\partial^3 f}{\partial x^2 \partial y}$, $\frac{\partial^3 f}{\partial y \partial x \partial y}$, $\frac{\partial^3 f}{\partial x \partial y^2}$, $\frac{\partial^3 f}{\partial y^3}$.
- 2^n n^{th}*-order partial derivatives.*

The reason for that is that any partial derivative is in itself a function depending on 2 variables for which we can define 2 partial derivatives $\frac{\partial}{\partial x}$ *and* $\frac{\partial}{\partial y}$.

Proposition 1. *Let $f(x,y)$ be a function of two variables. Suppose that the second-oder partial derivatives* $\frac{\partial^2 f}{\partial x \partial y}$ *and* $\frac{\partial^2 f}{\partial y \partial x}$ *are continuous. Then we have:*

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

4.1 Example

- $f(x,y) = x^2y^3$, $\frac{\partial f}{\partial x} = 2xy^3$, $\frac{\partial f}{\partial y} = 3x^2y^2$,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(2xy^3) = 6xy^2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(3x^2y^2) = 6xy^2.$$

4.2 Evaluation at a point

We can evaluate the partial derivatives at a point in the same way we do it for any function:

Suppose $f(x,y) = x^3\cos(2y)$, then $\frac{\partial^2 f}{\partial x \partial y} = -6x^2\sin(2y)$. We can evaluate these functions at $(2, \frac{\pi}{4})$:

$$f(2, \frac{\pi}{4}) = 2^3\cos(2\frac{\pi}{4}) = 0,$$

$$\frac{\partial^2 f}{\partial x \partial y}(2, \frac{\pi}{4}) = -6(2)^2\sin(2\frac{\pi}{4}) = -24.$$

5 Total differential

Definition 3. *Let $f(x,y)$ be a function of 2 variables. Then, we define the total differential of f by the following expression:*

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

5.1 Interpretation

The previous expression is not a function or a number, it is just a formal expression relating *infinitesimal (infinitely small) changes in the variables x and y, denoted by dx and dy respectively, to* infinitesimal changes in f denoted by df.

For a function of 3 variables, $g(x,y,z)$, the total differential would be:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

6 Application

The partial derivatives correspond to the rate of change of a function when one variable changes.

Let $f(x, y)$ be a function of 2 variables. Suppose that we change x from its original value by δx. Then $f(x, y)$ will change by δf such that:

$$f(x + \delta x, y) = f(x, y) + \delta f,$$

which means that:

$$\delta f = f(x + \delta x, y) - f(x, y).$$

If we divide both sides by δx (the quantity by which x changed), we get:

$$\frac{\delta f}{\delta x} = \frac{f(x + \delta x, y) - f(x, y)}{\delta x}.$$

On the right-hand side, we can identify clearly the rate of change of f with respect to x which limit as δx goes to 0 is $\frac{\partial f}{\partial x}$.

It follows that the limit of $\frac{\delta f}{\delta x}$ as δx goes to 0 is $\frac{\partial f}{\partial x}$. We can deduce then, that if δx is small enough we will be close to this limit, i.e. if δx *is small enough*, then

$$\frac{\delta f}{\delta x} \simeq \frac{\partial f}{\partial x},$$

or equivalently:

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x.$$

An analogous result holds when we replace x by y.

Now suppose that both x and y change, then we will have:

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y,$$

which means that the value of f will be affected by both changes with the corresponding rates.

We recognize in the latter formula a similarity with the total differential. There is however a substantial difference, in the total differential we have a rigorous equality and the changes are infinitesimal (infinitely small), while in the above formula, we have an approximate equality and the changes are finite (small but finitely small).

6.1 Example

Consider a triangle ABC of which we measure the lengths b and c and the angle A (in radian). Suppose the measures are $120\,\mathrm{m}$, $85\,\mathrm{m}$, and $\pi/6$.

The area S of the triangle is then given by the formula:

$$S = \frac{1}{2} bc \sin(A),$$

which for our measures gives $S = 2550\,\mathrm{m}^2$.

Now suppose that our measures were not exact. More precisely, we suppose that when we measure b and c we have an error that is of at most $0.1\,\mathrm{m}$ and when we measure A we have an error of at most $\pi/200$.

This means that the exact value of b cannot be known, but we know from our measure that this exact value stands between $120 - 0.1 = 119.9$ and $120 + 0.1 = 120.1\,\mathrm{m}$.

Since the exact value of S is not accessible (because of the inevitable errors on the measurements), we would like to have an interval to which S belongs.

This can be done using partial derivatives. In fact, S is a function of 3 variables b, c, and A. Since the errors are small, we can write:

$$\delta S = \frac{\partial S}{\partial b}\delta b + \frac{\partial S}{\partial c}\delta c + \frac{\partial S}{\partial A}\delta A,$$

or

$$\delta S = \frac{1}{2} c \sin(A)\delta b + \frac{1}{2} b \sin(A)\delta c + \frac{1}{2} bc \cos(A)\delta A.$$

The maximal error is obtained when we sum the absolute values of all maximal errors:

$$\delta S_{\max} = \left|\frac{1}{2} c \sin(A)\right| \delta b_{\max} + \left|\frac{1}{2} b \sin(A)\right| \delta c_{\max} + \left|\frac{1}{2} bc \cos(A)\right| \delta A_{\max}.$$

We obtain $\delta S_{\max} = 74.5\,\mathrm{m}^2$. It follows that the exact area S_{exact} satisfies:

$$2475.5\,\mathrm{m}^2 = S - \delta S_{\max} \leq S_{\text{exact}} \leq S - \delta S_{\max} = 2624.5\,\mathrm{m}^2.$$

7 Case of a compound function

Let f be a function depending on two variables x and y. Suppose that the variables x and y depend on a third variable t, i.e. $x = x(t)$ and $y = y(t)$.

We define the function g depending on the variable t as:

$$g(t) = f\big(x(t), y(t)\big).$$

To determine the derivative g' of g, we first write the total differential of f:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Then,

$$\begin{aligned} g'(t) &= \frac{dg}{dt} \\ &= \frac{df\big(x(t), y(t)\big)}{dt} \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \end{aligned}$$

Example

Suppose that in a rectangle of sides a and b the first side is increasing at a speed of $2\,\mathrm{m\,s^{-1}}$ and the second is decreasing at a speed of $4\,\mathrm{m\,s}$. What is the rate of change of the diagonal d of the triangle?

We can think of the length a, b, and d as functions of the time t. Then :

$$d(t) = \sqrt{a(t)^2 + b(t)^2}.$$

If we define f as $f(a, b) = \sqrt{a^2 + b^2}$, then $\frac{\partial f}{\partial a} = \frac{a}{\sqrt{a^2+b^2}}$ and $\frac{\partial f}{\partial b} = \frac{b}{\sqrt{a^2+b^2}}$. We deduce then, that the rate of change of the length of diagonal is given by:

$$\begin{aligned} d'(t) &= \frac{\partial f}{\partial a}\frac{da}{dt} + \frac{\partial f}{\partial b}\frac{db}{dt} \\ &= \frac{a}{\sqrt{a^2 + b^2}}a'(t) + \frac{b}{\sqrt{a^2 + b^2}}b'(t) \\ &= \frac{2a - 4b}{\sqrt{a^2 + b^2}}. \end{aligned}$$

8 Past exam questions

2006-7 (a) Find all second order partial derivatives of the following functions: $w = x^2 + xy^2 + xyz^2$; $z = e^{x^2 y}$. (b) The area of a rhombus is calculated using the formula $A = b^2 \sin C$. Using the measured values of $b = 4\,\mathrm{m}$ and $C = 45°$ respectively. Find using partial differentiation the maximum error in the area as calculated if there is a maximum error of $0.3\,\mathrm{cm}$ in the measurement of b and $0.5°$ in the measurement of C.

2005-6 (a) Find all second order partial derivatives of the following functions: $z = \frac{x^2}{y+1}$; $w = \cos(2xy)$. (b) The pressure, P, of an ideal gas is calculated from the formula $P = \frac{kT}{V}$ where T is the temperature $T = 20\,\mathrm{K}$[1] and $V = 1000\,\mathrm{cm}^3$. If the maximum error in T is $0.05\,\mathrm{K}$ and in V is $2\,\mathrm{cm}^3$, find, using partial differentiation and taking $k = 1$ the maximum error in P as calculated.

[1] It actually says $-20\,\mathrm{K}$ in the question but that violates the third law of thermodynamics.