Kinematics

MS4414 Theoretical Mechanics

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Contents

1 Recap

Vectors, scalars, integrals and derivatives.

Khan Academy
Vectors introduction.

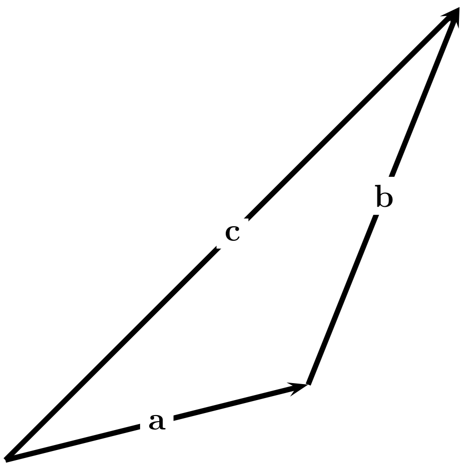
Khan Academy
Vector components.

1.1 Vectors and Scalars

A **scalar** quantity has a [magnitude] only. A **vector** quantity has both [magnitude] and [direction].

Quantity	Type
10	Scalar
$(10, 10)$	Vector
$54\,\mathrm{K}$	Scalar
$10\,\mathrm{m}$ north	Vector
$10\mathbf{e}_x + 1\mathbf{e}_y$	Vector
$32\,\mathrm{kg}$	Scalar

Vector addition

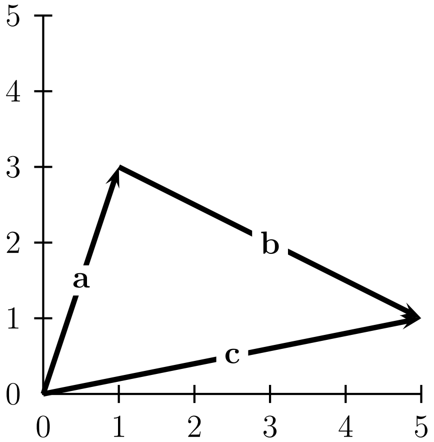


The sum of two vectors $\mathbf{c} = \mathbf{a} + \mathbf{b}$ is obtained by adding them nose to tail. In terms of components

$$c_x = a_x + b_x$$

$$c_y = a_y + b_y$$

$$c_z = a_z + b_z$$



If $\mathbf{a} = (1, 3)$, $\mathbf{b} = (4, -2)$ and $\mathbf{c} = \mathbf{a} + \mathbf{b}$ the components of $\mathbf{c}$ are

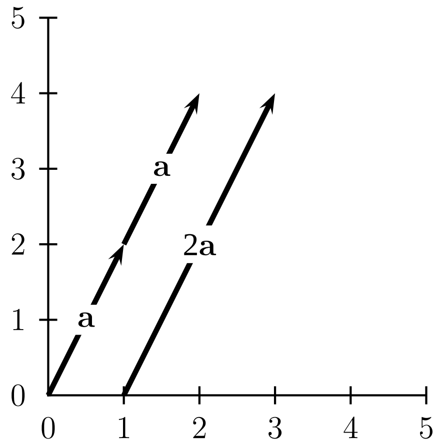
$$c_x = 1 + 4 = 5$$

$$c_y = 3 - 2 = 1$$

Multiplication of a vector and a scalar Multiplication of a vector $\mathbf{a}$ by a scalar λ gives a vector $\mathbf{b} = \lambda\mathbf{a}$ with the same direction as $\mathbf{a}$ and a magnitude λ times the magnitude of $\mathbf{a}$.

$$b_x = \boxed{\lambda a_x}$$

$$b_y = \boxed{\lambda a_y}$$



$\mathbf{a} = (1, 2)$ and $\lambda = 2$, $\mathbf{b} = \lambda\mathbf{a}$.

$$b_x = \boxed{2 \times 1 = 2}$$

$$b_y = \boxed{2 \times 2 = 4}$$

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Dot product.

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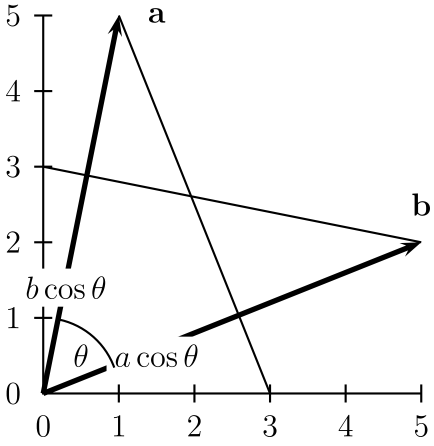
Dot product
properties.

Khan Academy
Dot product
in Physics.

Khan Academy
Cross product.

Scalar Product

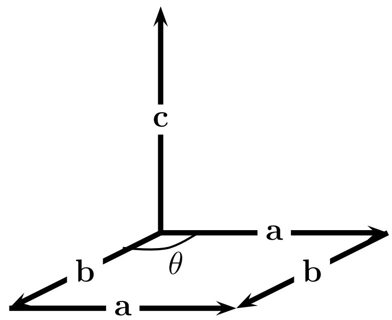


The scalar product of $\mathbf{a}$ and $\mathbf{b}$ is a scalar with magnitude $ab\cos\theta$. The projection of the vector $\mathbf{a}$ along the direction of the vector $\mathbf{b}$ has length $a\cos\theta$. The projection of the vector $\mathbf{b}$ along the direction of $\mathbf{a}$ is $b\cos\theta$.

If the dot product of two vectors is zero, the vectors are orthogonal. In terms of the components of the vectors the dot product is given by

$$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y$$

Cross Product



The cross product of two vectors $\mathbf{c} = \mathbf{a}\times\mathbf{b}$ is a vector orthogonal to the plane formed by the two vectors and with magnitude equal to the the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$. This area is $ab\sin\theta$

If two (nonzero) vectors have a zero cross product they are parallel.

The components of the cross product are given by

$$c_x = \boxed{a_y b_z - a_z b_y}$$

$$c_y = \boxed{a_z b_x - a_x b_z}$$

$$c_z = \boxed{a_x b_y - a_y b_x}$$

What do you get if you cross a sheep with a goat?

Khan Academy Derivative.

1.2 Differentiation

The derivative of a function $f(t)$ is defined as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \boxed{\lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t}}$$

The derivative gives the gradient of a graph of $f(t)$ plotted against t. The derivative of $f(t)$ is the rate of change of f with respect to t.

Function	Derivative
5	0
t	1
t^n	nt^{n-1}
$\cos t$	$-\sin t$

Khan Academy
Integration.

1.3 Integration

The opposite operation to differentiation is integration.

$$\frac{\mathrm{d}F}{\mathrm{d}t} = f(t)$$

$$\boxed{\int f(t)\ \mathrm{d}t} = F(t) + C$$

$$\boxed{\int_a^b f(t)\ \mathrm{d}t} = F(b) - F(a)$$

Khan Academy
Definite integration.

The integral of $f(t)$ gives the area under the curve of f plotted against t. Integrals can be definite or indefinite. An indefinite integral always contains a constant because the derivative of a constant is zero.

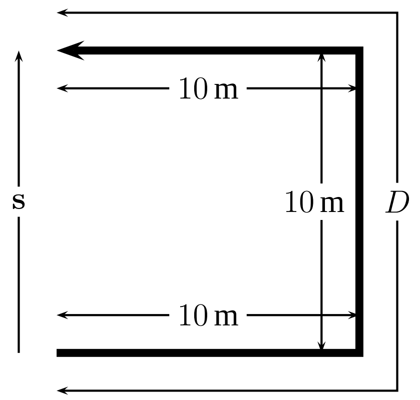
Function	Indefinite Integral
5	$5t + C$
t	$\frac{1}{2}t^2 + C$
t^n	$\frac{1}{n+1}t^{n+1} + C$
e^t	$e^t + C$

Definite Integral	Result
$\int_0^2 5\,\mathrm{d}t$	$[5t]_0^2 = 10$
$\int_{-2}^2 t^2\,\mathrm{d}t$	$\left[\frac{1}{3}t^3\right]_{-2}^2 = \frac{16}{3}$
$\int_0^\pi \sin t\,\mathrm{d}t$	$[-\cos t]_0^\pi = 2$
$\int_1^2 t^{-2}\,\mathrm{d}t$	$\left[-\frac{1}{t}\right]_1^2 = \frac{1}{2}$

Khan Academy
Vectors and scalars, distance and displacement.

2 Distance and displacement

A person (far from either pole) walks $10\,\mathrm{m}$ east, $10\,\mathrm{m}$ north and $10\,\mathrm{m}$ west. How far has he travelled? There are two answers to this question: they are $30\,\mathrm{m}$ and $10\,\mathrm{m}$. Both answers are correct but sometimes one is more useful than the other.



The **distance**, D, is the total length of the path the person has travelled, which could be measured by multiplying the total number of steps the person took by the length of their stride. The **displacement**, s, of the person is vector from their initial point the their final point.

Displacement is a vector quantity because it has both a [magnitude] (in the example above [10 m]) and a [direction] (in the example above [north]). Distance is a scalar quantity, because while it has a [magnitude] (in the example above [30 m]), it has no unique [direction].

Khan Academy
Velocity calculation.

3 Speed and Velocity

From distance and displacement we can describe two different rates of change. The speed is the rate of change of distance with respect to time. The velocity is the rate of change of displacement with respect to time. [Speed] is a scalar quantity. [Velocity] is a vector quantity. In the language of calculus:

- Speed is the [derivative] of distance with respect to time.
- Velocity is the [derivative] of displacement with respect to time.

Since integration is the opposite of [differentiation] we also have that:

- Distance is the integral of speed with respect to time.
- Displacement is the integral of velocity with respect to time.

$$U = \frac{\mathrm{d}D}{\mathrm{d}t}$$

$$v = \frac{\mathrm{d}s}{\mathrm{d}t}$$

$$D = \int U \,\mathrm{d}t$$

$$s = \int v \,\mathrm{d}t$$

Khan Academy Velocity integral.

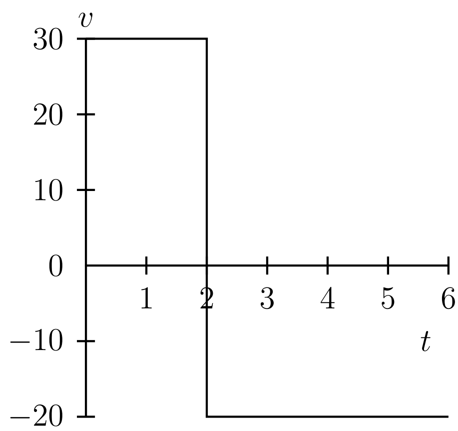
where D is distance, s is displacement, U is speed, v is velocity.

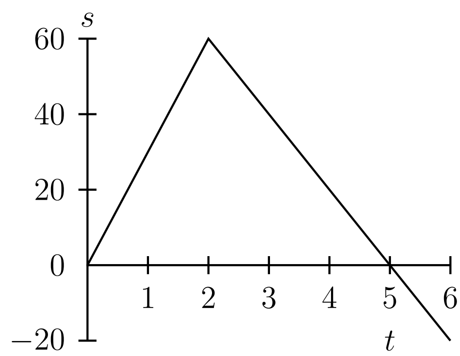
Khan Academy Example 1.

Khan Academy Example 2.

Worked Example A cyclist sets out with velocity $30\,\mathrm{km\,hr^{-1}}$ and cycles for two hours. He then cycles for four hours in the opposite direction at $20\,\mathrm{km\,hr^{-1}}$.

1. Draw graphs of the velocity and displacement of the cyclist vs. time.
2. Find how far the cyclist ends up from his starting point.
3. What is the total distance travelled by the cyclist and his mean velocity?





Cyclist ends up -20 km from his starting point. Total distance traveled: $60 + 80 = 140$ km. Average velocity: $(-20\,\text{km})\,/\,6\,\text{hr} = 3.33\,\text{km}\,\text{hr}^{-1}$. Average speed (not asked for): $(140\,\text{km})\,/\,(6\,\text{hr}) = 23.33\,\text{km}\,\text{hr}^{-1}$

Microsoft Hiring Question Four dogs are positioned at the corners of a square with edge length $100\,\text{m}$. Each dog runs towards his anticlockwise neighbour with speed $10\,\text{m}\,\text{s}^{-1}$. How long does it take the dogs to catch up with each other?

Resolve the velocity of each dog into polar coordinates. The radial component is $(-10\,\text{m}/\,\text{s}^{-1})\,/\sqrt{2}$. The distance the dogs have to travel in order to meet is $(10\,\text{m})\,/\sqrt{2}$. This takes $10\,\text{s}$.

Khan Academy Acceleration.

4 Acceleration

One final quantity important in mechanics is **acceleration**. Acceleration is important because the acceleration of a particle of fixed mass can be related to the forces imposed on it by Newtons Laws of motion (which will be covered later).

Acceleration is the rate of change of velocity. Thus Acceleration is the first derivative of velocity with respect to time. Acceleration is also the second derivative of displacement with respect to time.

Acceleration is a vector quantity.

$$a = \frac{\mathrm{d}}{\mathrm{d}t} v$$

$$a = \frac{\mathrm{d}^2}{\mathrm{d}t^2} s$$

Khan Academy Constant acceleration.

5 Constant Acceleration

If the acceleration is constant then the equations

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}$$

and

$$a = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

can be integrated once

$$v = \boxed{v_0 + at}$$

and a second time

$$s = \boxed{s_0 + v_0 t + \tfrac{1}{2} a t^2}$$

where the initial $(t = 0)$ velocity is v_0 and position is x_0.

An important constant acceleration case is given by the Earth's gravitational field (close to the Earth's surface) in the absence of air resistance. This acceleration is $g = 9.81\,\mathrm{m\,s^{-2}}$ and directed downwards. (Often g is taken as $10\,\mathrm{m\,s^{-2}}$ to make calculations simpler.)

5.1 One dimensional motion under gravity

Khan Academy 1d motion.

In one dimension we have $s = y$, the height of the particle, and $a = -g$ the acceleration due to gravity.

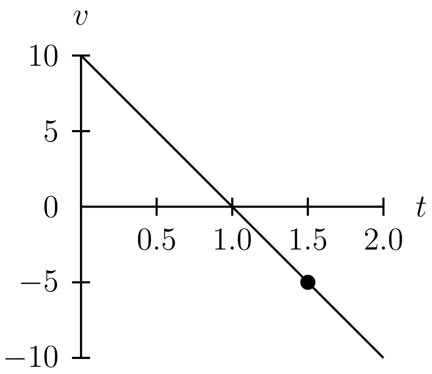
$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 y}{\mathrm{d}t} = \boxed{-g}$$

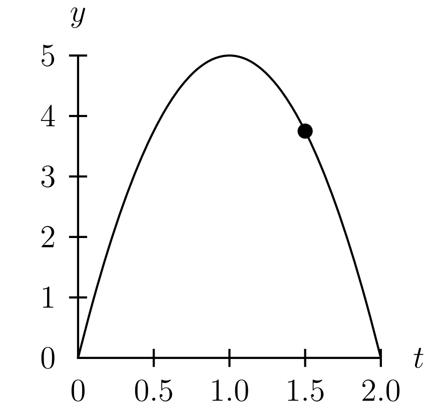
$$v = \frac{\mathrm{d}y}{\mathrm{d}t} = \boxed{v_0 - gt}$$

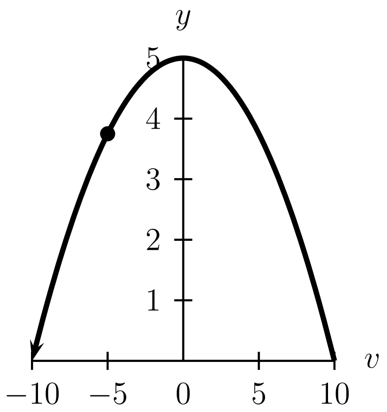
$$y = \boxed{y_0 - v_0 t - \tfrac{1}{2} g t^2}$$

Khan Academy 1d motion example.

Worked example Take $g = 10\,\mathrm{m\,s^{-2}}$ and neglect air resistance. A stone is thrown up in the air with velocity $v_0 = 10\,\mathrm{m\,s^{-1}}$. What is the maximum height reached by the stone? How long does it take the stone to fall to Earth? How high is the stone when its velocity is $-5\,\mathrm{m\,s^{-1}}$? How long does it take to reach this state? Draw graphs of the velocity v and height y as a function of time. Draw the parametric curve of y (y-axis) against v (x-axis): indicate the direction of increasing time with an arrow.







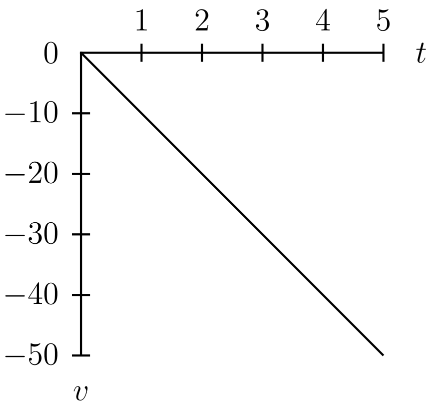
Maximum height reached by stone: $4\,\mathrm{m}$.

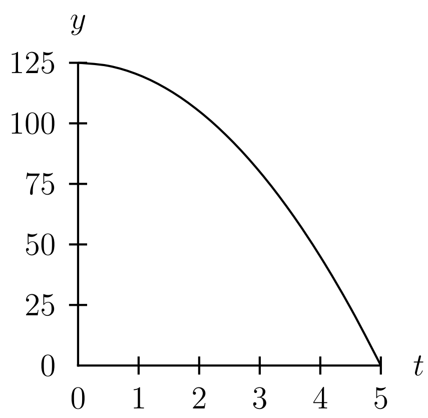
Time taken for stone to return to Earth: $2\,\mathrm{s}$.

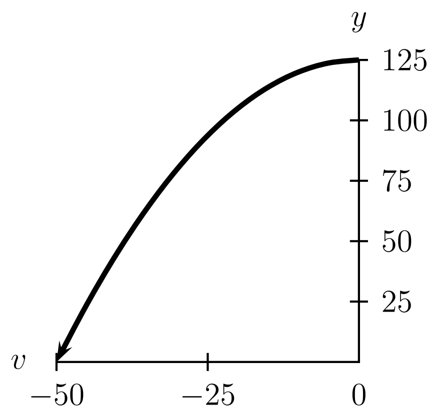
The stone is $3.75\,\mathrm{m}$ high when its velocity is $-5\,\mathrm{m\,s^{-1}}$. It takes $1.5\,\mathrm{s}$ to attain this velocity.

Khan Academy
Height from time.

Worked example Neglect air resistance and take $g = 10\,\mathrm{m\,s^{-2}}$. A stone is dropped from a tower of height H and hits the ground $5\,\mathrm{s}$ later. How high is the tower? How fast was the stone travelling when it hit the ground? Draw graphs of the height and velocity of the stone as a function of time. Draw the parametric curve of y against v, using arrows to indicate the direction of increasing time.







Velocity of stone when it hits the ground: $-50\,\mathrm{m\,s^{-1}}$. Height of tower: $125\,\mathrm{m}$.

Catching Dropped Baseballs

- 1938. Frankie Pytlak and Hank Helf catch baseballs dropped from top of $213\,\mathrm{m}$ high building. (World record. Missed ball bounced up to 13th floor.)
- 1939. Joe Sprintz tries to catch ball thrown from blimp at $244\,\mathrm{m}$. Ball drove hand into face breaking jaw, five teeth and knocking him unconscious. . . and he dropped the ball.
- 1916 Willie Robinson tried to catch baseball thrown from plane at $122\,\mathrm{m}$. Mean friend substituted a red grapefruit, which exploded on impact. Robinson: ‘It broke me open! I’m covered in blood.’

Jearl Walker, Flying circus of physics.

5.2 Two dimensional motion under gravity

The equations derived above are just as valid for vectors

$$\boxed{\frac{\mathrm{d}^2\mathbf{s}}{\mathrm{d}t^2}} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{g}$$

$$\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} = \mathbf{v} = \boxed{\mathbf{v}_0 + \mathbf{g}t}$$

$$\mathbf{s} = \boxed{\mathbf{s}_0 + \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{g}t^2}$$

Note the $+$ signs in the terms containing g. The $-$ sign is in the components of g

$$\mathbf{g} = \begin{pmatrix} g_x \\ g_y \end{pmatrix} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

Writing out the above equations in component form

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}t^2} \begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} 0 \\ -g \end{pmatrix} \quad (1)$$

Integrating if the position at $t = 0$ is $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ and the velocity at $t = 0$ $\begin{pmatrix} v_{x0} \\ v_{y0} \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \boxed{\begin{pmatrix} x_0 + v_{x0}t \\ y_0 + v_{y0}t - \frac{1}{2}gt^2 \end{pmatrix}} \quad (2)$$

Exam Question 2007 A stone is projected (under gravity, with air friction neglected) with velocity v_0, at an angle α to the horizon, towards a 'step' of height H:

- Assuming that the stone goes over the step, calculate y_{max} (the maximum height of the stone's trajectory) and x_H (the x-coordinate of the point where it hits the ground).
- Determine for which v_0 the stone would go over the step.

$$v_x = v_0 \cos\alpha$$
$$v_y = v_0 \sin\alpha - gt$$

Maximum height reached when $v_y = 0$ at t_m.

$$t_m = \frac{g}{v_0 \sin\alpha}$$

$$x = v_0 t \cos\alpha$$
$$y = v_0 t \sin\alpha - \tfrac{1}{2}gt^2$$
$$y_m = \frac{v_0^2}{2g} \sin\alpha\,(1 - \sin\alpha)$$

Hits the ground at time t_H

$$t = \frac{v_0 \sin\alpha}{g} + \frac{1}{g}\sqrt{v_0^2 \sin^2\alpha - 2gH}$$

Stone goes over the step if

$$v_0 > \sqrt{\frac{2gH}{\sin\alpha\,(1-\sin\alpha)}}$$