Gravity

MS4414 Theoretical Mechanics

William Lee

Contents

1 Introduction

When the Astronauts of Apollo 8 were asked who was driving their spacecraft they replied, "I think Isaac Newton is doing most of the driving now".

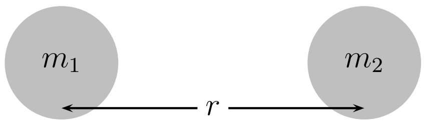
We already know Newton's laws of motion, once we know Newton's law of gravity you will know all the fundamental equations you need to pilot a spaceship (provided you drive at less than $\frac{1}{10}$th of the speed of light and stay away from neutron stars and black holes). All the other details of the motion of planets, asteroids, comets, satellites and spaceships can be worked out from those fundamental equations (although it's not easy). You can understand the tides, how a gravitational slingshot works and plot the best course for a manned mission from the Earth to Mars.

2 Newton's Law of Gravitation

Newton's law of gravitation states that the attraction between tow bodies with masses m_1 and m_2 separated by a distance r is given by

$$F = -\frac{Gm_1m_2}{r^2}$$

The minus sign indicates that the gravitational force is constant.



3 Close to the surface of the earth.

Close to the surface of the earth I can write the force on an object at height h relative to the Earth's surface as

$$F = -\frac{GMm}{(R+h)^2}$$

I can rearrange this into the form

$$F = -\frac{GMm}{R^2}\left(\boxed{1 + h/R}\right)^{-2}$$

and, provided $\boxed{h/R}$ is small, expand the bracket using a Taylor series

$$f(R+h) = f(R) + \boxed{h}\, f'\left(\boxed{R}\right)$$

$$F = -\frac{GMm}{R^2} + \frac{GMmh}{R^3} + \dots$$

Or in terms of g

$$F = \boxed{-gm} + \boxed{g\frac{h}{R}}$$

where

$$g = \boxed{\frac{GM}{R^2}}$$

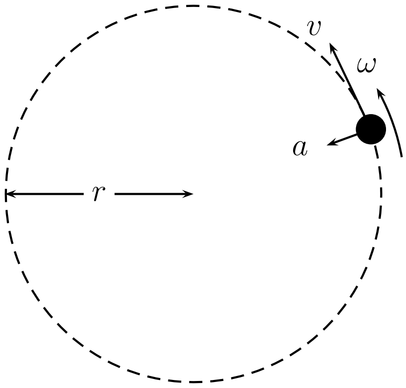
The gravitational field of the Earth can be treated as constant if $h \ll \boxed{R}$.

4 Orbits

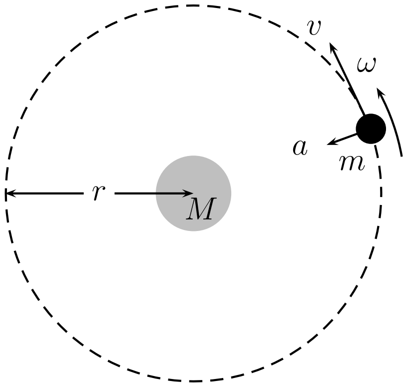
As we have seen previously, a body undergoing circular motion at radius r with angular velocity ω, an linear velocity $v = \boxed{r}\,\omega$, has an acceleration a

$$a = \boxed{\frac{v^2}{r}} = \boxed{\omega r^2}$$

directed radially inwards.



For a particle of constant mass, Newton's second law states $\boxed{F = ma}$, in other words, this acceleration must be caused by a force. Suppose the particle has mass m and is orbiting a body of mass M.



Using Newton's law of gravitation the force on the particle due to the gravitational attraction of the two masses is

$$F = \boxed{\frac{GMm}{r^2}}$$

and directed inwards towards mass M. This is equal to the acceleration of circular motion.

$$F = \boxed{\frac{GMm}{r^2}} = \boxed{mv^2/r}$$

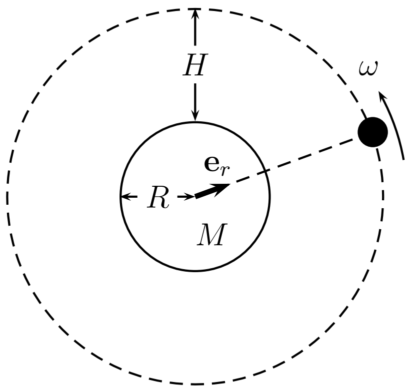
Solving for v we get

$$v = \boxed{\sqrt{GM/r}}$$

Or in terms of ω

$$\omega = \boxed{\frac{v}{r}} = \boxed{\sqrt{\frac{GM}{r^3}}}$$

Exam question A satellite of mass m is rotating about a planet of mass M and radius R with angular velocity ω. The satellite is at height H. We suppose that the satellite is subject only to gravitational attraction of the planet.



1. Find the acceleration $\mathbf{a}$ of the satellite in terms of R, H, ω, and the unit vector $\mathbf{e}_r$ oriented from the centre of the planet to the satellite.
2. By applying Newton's second law and Newton's law of gravity, express the angular velocity ω in terms of M, R, H, and the gravitational constant $\mathcal{G}$.

5 Gravitational Potential energy

Consider two masses m_1 and m_2 separated by an infinite distance. This is taken to be the zero of gravitational potential energy. Hold mass m_2 fixed in place and let mass m_2 move freely (this isn't necessary, it just makes the maths a bit easier).

Using Newton's second law

$$m\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \boxed{-\frac{Gm_1m_2}{r^2}}$$

multiply through by $\frac{\mathrm{d}r}{\mathrm{d}t}$

$$m\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \boxed{-\frac{Gm_1m_2}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}}$$

and write as a total derivative

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}mv^2 - \boxed{\frac{Gm_1m_2}{r}}\right]$$

Integrate and call the constant of integration E

$$E = \frac{1}{2}mv^2 - \boxed{\frac{Gm_1m_2}{r}}$$

The kinetic energy is

$$\text{KE} = \frac{1}{2}mv^2$$

The gravitational potential energy is

$$\text{PE} = \boxed{-\frac{Gm_1m_2}{r}}.$$

Exam Question Two spherical objects of radii $R_{1,2}$ and masses $m_{1,2}$, are attracted to each other through gravity. The initial velocities of the objects are zero, the initial distance separating them is infinitely large. Find their velocities when they collide. (2007 resit.)

Conservation of momentum

$$\boxed{0} + \boxed{0} = \boxed{m_1v_1} + \boxed{m_2v_2}$$

Initial potential energy $\boxed{0}$. Initial kinetic energy $\boxed{0} + \boxed{0}$. Final potential energy $\boxed{-\frac{Gm_1 \ldots}{R_1 + \ldots}}$

Final kinetic energy $\boxed{\frac{1}{2}m_1v_1^2} + \boxed{\frac{1}{2}m_2v_2^2}$.

Conservation of energy

$$0 = \boxed{-\frac{Gm_1m_2}{R_1 + R_2}} + \boxed{\frac{1}{2}m_1v_1^2} + \boxed{\frac{1}{2}m_2v_2^2}$$